Theory of Special Relativity

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Chapter 1

Introduction

1. Inertial frame (Galilean frame):

Newton's first law of inertia states that a body at rest remains at rest and a body in motion continues with constant velocity in a straight line unless an external force is applied to it.

The above statement has do meaning, i.e., the motion of a body has no meaning unless it is described with respect to some well defined co-ordinate system or frame of reference relative to which the velocity of the body is measured. This led Newton to introduce the idea of absolute space, which represents the system of reference relative to which the motion of a body can be defined.

The question naturally arises how to find an absolute system of reference. The choice of frame of reference should be such that the laws of nature may become simpler when they are expressed in terms of such frames of reference.

There are two types of frames of reference:

(i) Accelerated frame of reference.

(ii) Unaccelerated frame of reference.

We shall have to choose unaccelerated frame of references. Since in such frames of references all the laws of mechanics

preserve the same form when they are expressed in terms of any of these frames of references. For these frames are moving with uniform velocity.

Consider a co-ordinate system relative to which the co-ordinates of a body in motion are x , y , z. The co-ordinates x , y , z are functions of time t. Since any force does not act the body, i.e., it is moving which constant velocity.

$$\frac{d^2x}{d\,t^2} = 0 \quad , \quad \frac{d^2y}{d\,t^2} = 0 \quad , \quad \frac{d^2z}{d\,t^2} = 0$$

$$\frac{d\,x}{d\,t} = u \quad , \quad \frac{d\,y}{d\,t} = v \quad , \quad \frac{d\,z}{d\,t} = 0$$

u , v , w being velocity components in x , y , z directions respectively. This is Newton's first law of inertia.

Such a co-ordinate system is called inertial frame. Thus an inertial frame of reference is one in which Newton's first law of motion holds good.

2. Galilean Transformations:

The consequences of research work of Galileo on the motion of the projectile led him to formulate transformations, which later on, were called after his name "Galilean transformations". These are used to describe the motions, which are observed by two observers in two different inertial frames. His two main results are as follows:

(i) The motion of a particle projected at any angle may be derived from the motion of the panicle thrown vertically upward.

(ii) If a particle is thrown straight up from a cart, which is moving with uniform speed, the observer on the cart may see the particle moving up and down. But the motion observed by an observer on the ground may be described by superimposing the motion of the cart into that of the projectile.

Consider two frames S and S' of references one at rest and the other is moving with uniform velocity v. Let O and O' be the observers situated at the origins of S and S' respectively. They are observing the same event at any point P. let the two frames be parallel to each other, i.e., X'-axis is parallel to X-axis, Y'-axis is parallel to Y-axis, Z'-axis is parallel to Z-axis. Let the coordinates of P be (x, y, z, t) and (x', y', z', t') relative to origins O and O' respectively

The choice of the origins of two frames is such that their origins coincide at time $t = 0$, $t' = 0$

Case I. Let the frame S' have the velocity v only in X'-direction.

Then O' has velocity v only along X'-axis. The two systems can be combined to each other by the following equations

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \qquad (1)$$

Case II. Let the frame S' have velocity v along any straight line in any direction such that $v = i\,v_x + j\,v_y + k\,v_z$.

After a time t , the frame S' separated from S by distances tv_x , tv_y , tv_z along X , Y , Z axes respectively. Then the two systems can be related by the following equations:

$$\left.\begin{aligned} x' &= x - tv_x \\ y' &= y - tv_y \\ z' &= z - tv_z \\ t' &= t \end{aligned}\right\} \qquad (2)$$

Transformations (1) and (2) are called Galilean transformations.

Theorem 1: Invariance of Newton's Law. To prove that Newtonian fundamental equations are invariant under Galilean transformations.

Proof. We prove the assertion by taking Newton's second law of motion. In the absolute system of co-ordinates, a particle acted upon by a force F has an acceleration $\frac{d^2x}{d\,t^2}$ such that:

$$F = m\frac{d^2x}{d\,t^2} \qquad (1)$$

In Galilean frame of reference,

$$x' = x - v\,t \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = t$$

$$\therefore \frac{d\,x'}{d\,t} = \frac{d\,x}{d\,t} - v \qquad d\,t' = d\,t$$

$$\therefore \frac{d\,x'}{d\,t'} = \frac{d\,x}{d\,t} - v \qquad \therefore \frac{d^2x'}{d\,t'^2} = \frac{d^2x}{d\,t^2} \tag{2}$$

In Newtonian mechanics, forces and masses are absolute quantities so that:

$$m' = m \quad , \quad F' = F \tag{3}$$

Putting the values from (2) and (3) in (1):

$$F' = m'\frac{d^2x'}{d\,t'^2} \tag{4}$$

Comparing (1) and (4), we can say that Newton's second law of motion is invariant under Galilean transformation.

3. Electrodynamics:

Classical mechanics is that branch of physics, in which the forces acting between two moving charges depends upon the distance between them and the direction of the force is along the straight line joining the charges.

Electrodynamics is that branch of physics, in which the force acting between two moving charges depends upon the distance

between them and their velocities. Also the direction of acting force is not a straight line joining the charges.

This is the fundamental difference between classical mechanics and electrodynamics. Therefore the basic concepts of classical mechanics are not applicable to electrodynamics. The extension of the principles of relativity to electrodynamics results in the four fundamental equations of Maxwell's for empty space by performing the experiments in two different inertial frames. According to Maxwell; "Electromagnetic waves propagate in empty space with a uniform velocity $c = 3 \times 10^{10}$ cms. /second. Light waves are electromagnetic waves and the velocity of light in vacuum is independent of the state of motion of the source of light and is equal to a constant value, $c = 3 \times 10^{10}$ cms. /second.

Then the velocity of light must have the constant value c in all inertial systems independent of the motion of the source of light.

But this is contrary to the classical theory, as we have seen in Galilean transformations that the velocity of a particle has a lower value in the system S' in the direction of the velocity of the particle than a system S at rest. Thus the constancy of velocity of light and the principle of relativity in classical mechanics are contrary to each other. Therefore we have to

revise our ordinary concept of space-time if we accept the principle of relativity in an electromagnetic field.

Remark. It is impossible to transmit any signal with a velocity greater than the velocity of light, i.e., c is an upper limit for the velocity with which a particle can move

4. Fizeau's Experiment:

The aim of this experiment was to detect the absolute velocity of the earth through ether. The experimental arrangement is as shown in the Fig. 1.

A beam of light from a source S is divided by half silvered mirror P, placed at 45° into transmitted part (1) and reflected part (2). The transmitted ray (1) passes through water running in opposite directions and is then

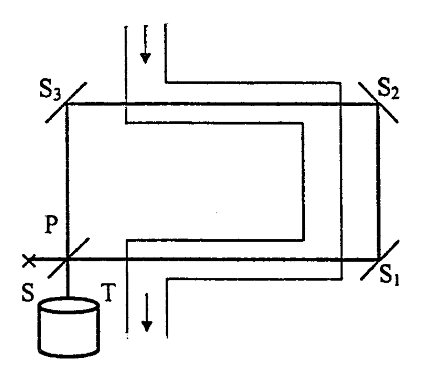


Fig. 1.

reflected by mirror S_1 then by another mirror S_2 and then again passes through water running in opposite directions and after reflection from mirror S_3 falls on P, a part of it is transmitted through P and reaches the telescope T

The reflected ray (2) at P goes along P S_3 S_2 S_1 P passing through water along S_3 S_2 and S_1 P running in the same direction and is then reflected by P and enters the telescope T where it interferes with the ray (1) and the interference fringes are observed. The time difference between the two rays is:

Time difference:

$$= \frac{l}{\frac{c}{\mu} + \alpha v - u\left[1 - \frac{1}{\mu^2}\right]} - \frac{l}{\frac{c}{\mu} + \alpha v - u\left[1 - \frac{1}{\mu^2}\right]}$$

Where l is the length of whole path on which light travels in water, u the velocity of water and $\alpha = \frac{1}{\mu^2}$ = constant.

Time difference:

$$= l\left[\frac{c}{\mu}\left\{1 - \frac{\alpha\mu v}{c} + \frac{\mu u}{c}\left(1 - \frac{1}{\mu^2}\right)\right\} - \frac{c}{\mu}\left\{1 - \frac{\alpha\mu v}{c} - \frac{\mu u}{c}\left(1 - \frac{1}{\mu^2}\right)\right\}\right]$$

$$= \frac{2l\mu^2 u}{c^2}\left(1 - \frac{1}{\mu^2}\right) = \frac{2lu}{c^2}\,(\mu^2 - 1)$$

The phase difference is obtained by dividing the expression by the period T of one oscillation. Let n be the frequency of the oscillation of light. Then n T = 1.

Phase difference:

$$= \frac{2lu}{c^2}\,(\mu^2 - 1)/T = \frac{2lu}{c^2}\,(\mu^2 - 1)/(1/n)$$

$$= \frac{2lun}{c^2} (\mu^2 - 1)$$

The shift in the position of the fringes observed by Fizeau experiment was in complete agreement with this formula. Since the phase difference is independent of v. So the theory fails to find velocity of earth through ether. It simply confirms the present result namely that the velocity of light in any medium of refractive index μ is not simply $\frac{c}{\mu}$, but $\frac{c}{\mu} \pm u \left[1 - \frac{1}{\mu^2}\right]$, where u is the velocity of medium. Thus so long as the medium is at rest, the ether inside is also at rest. But when the medium has a velocity u the ether inside it is drifted in the direction of u by an amount $\pm u \left[1 - \frac{1}{\mu^2}\right]$. The factor $\left[1 - \frac{1}{\mu^2}\right]$ is called Fresnel coefficient.

Thus the velocity of light relative to a material body is not constant, but it depends on the nature of medium and velocity of the medium itself through ether.

Mathematical theory: If the velocity of water is u and velocity of light in any medium of refractive index μ is c/μ then from, addition theorem of velocities,

$$V = \frac{u + v}{1 + \frac{uv}{c^2}} = \frac{u + (c/\mu)}{1 + \frac{c}{\mu} \cdot \frac{u}{c^2}} = \frac{u + \frac{c}{\mu}}{1 + \frac{u}{c\mu}} = \frac{c}{\mu}\left(1 + \frac{\mu u}{c}\right)\left[1 + \frac{u}{c\mu}\right]^{-1}$$

$$= \frac{c}{\mu}\left[1 + \frac{\mu u}{c}\right]\left[1 - \frac{u}{c\mu}\right], \text{ neglecting higher terms.}$$

$$= \frac{c}{\mu}\left[1 - \frac{u}{c\mu} + \frac{\mu u}{c} - \frac{u^2}{c^2}\right]$$

$$= \frac{c}{\mu}\left[1 - u\left(\frac{1}{c\mu} - \frac{\mu}{c}\right)\right] = \frac{c}{\mu} + u\left[1 - \frac{1}{\mu^2}\right]$$

5. Michelson and Morley experiment.

Michelson and Morley performed this important experiment in 1887 to determine the relative velocity of light with respect to earth. For this purpose the velocity of earth with respect to ether was essential.

The arrangement of the experiment consists in a monochromatic source S of light falling on a half silvered plate P_1 inclined at an angle 45° to the beam of light from S.

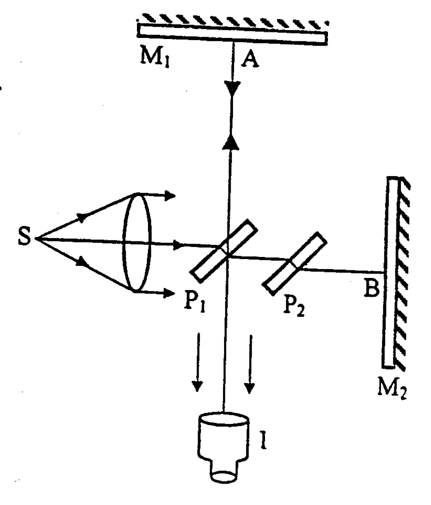


Fig. 2.

The plate P_1 being half silvered divides the beam of light into two, one being reflected perpendicular to its original direction and the other transmitted through it.

The reflected beam falls on a plane mirror M_1 at A normally and is reflected back along its own path, passing through P_1, it enters the telescope T. The other beam, transmitted through P_1, passes through a plate P_2 of thickness equal to that of P_1 and mounted parallel to it and is then reflected normally by a plane mirror M_2 at B along its own path.

The reflected beam after passing through P_2 falls upon P_1 and reflected to the telescope T. Both the mirrors M_1 and M_2 are highly polished to avoid multiple total internal reflection and are at equal distances from P_1, i.e., $P_1B = P_1A$.

The two beams reflected from the mirrors M_1 and M_2 enter the telescope and cause interference, which result interference fringes. The purpose of the plate P_2 is just to compensate the extra path traveled by the reflected beam, since it crosses the plate P_1 twice. Hence in order that the transmitted beam may have to travel an equal extra distance, the plate P_2 is inserted in the path, so that the two beams before entering the telescope may travel equal distances.

Now let us consider that the whole apparatus is moving with the velocity of earth towards right, i.e., along SP_1B, ether being regarded to be stationary. Due to the motion of the earth, the reflection of the ray by the mirror M_1 does not take place at A but at A' and similarly by M_2 at B' instead of B.

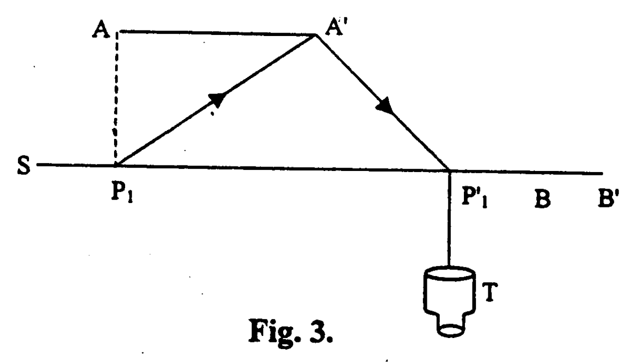


Fig. 3.

Due to the motion of the earth, the plate P_1 moves to P'_1 during this time and hence the interference of the two beams takes place in the telescope T. It is obvious that the lengths of the paths traveled by the reflected and transmitted beams are not equal in this case, the calculation will be as follows:

Calculation: Suppose

c = velocity of light w . r . t. ether.

v = velocity of earth w . r . t. ether, $P_1A = l$.

t_1 = time taken by the transmitted beam which travels the distance from P_1 to B' and then B' to P'_1.

t_2 = time taken by the reflected beam which travels from P_1 to A' and then A' to P'_1.

Since the mirror M_1 at A and mirror M_2 at B are placed at equal distance and therefore:

$$P_1A = P_1B = l$$

To calculate t_1: The apparatus together with the earth is moving with velocity v towards the right. Take the laboratory as standard of rest. It means that the apparatus may be supposed as situated in the stream of ether moving with velocity v towards the left. Velocity of light w . r . t. apparatus towards right.

= velocity of light w . r . t. ether + velocity of ether w . r . t. apparatus.

$= c + (-v) = c - v.$

Thus $c - v$ is the relative velocity of light along P_1B'.

Similarly $c + v$ is the relative velocity of light along $B'P_1$.

t_1 = time taken by the beam from P_1 to B' and B' to P_1.

$$= \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2} = \frac{2lc}{c^2}\left[1 - \frac{v^2}{c^2}\right]^{-1}$$

$$= \frac{2lc}{c^2}\left[1 + \frac{v^2}{c^2}\right] = \frac{2l}{c}\left[1 + \frac{v^2}{c^2}\right]$$

v being small in comparison to c, $\frac{v^2}{c^2}$ and higher powers of $\frac{v}{c}$ have been neglected.

To calculate t_2: For the path $P_1A'P_1$, when the beam travels from P_1 to A', the apparatus travels from A to A' and hence:

$$\frac{AA'}{P_1A'} = \frac{v}{c}$$

Hence $AA' = \frac{v}{c} P_1A'$

In $\Delta\, P_1AA' = P_1A^2 + A'A^2 = P_1A'^2$

Or $l^2 + \frac{v^2}{c^2} P_1A'^2 = P_1A'^2$

Or $P_1A' = \frac{l}{\sqrt{1-\frac{v^2}{c^2}}} = l\left[1-\frac{v^2}{c^2}\right]^{-1/2} = l\left[1+\frac{v^2}{2c^2}\right]$

t_2 = time taken by the beam from P_1 to A and A' to P_1.

= 2 time taken by the beam from P_1 to A.

$= 2\frac{P_1A'}{c}$ For time $= \frac{\text{distance}}{\text{velocity}}$

$= \frac{2l}{c}\left[1+\frac{v^2}{2c^2}\right]$

Since the relative distance w . r . t. the apparatus is l in journey towards B and backwards, hence the difference of two timings is:

$$\Delta\, t = t_1 - t_2 = \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right) - \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right)$$

$$= \frac{2l}{c}\left[1 + \frac{v^2}{c^2} - 1 - \frac{v^2}{2c^2}\right] = \frac{lv^2}{c^3}$$

When the apparatus is rotated through 90°, the two systems will be in exchanged positions and hence:

$$t_1 = t_2 \ , \ \bar{t}_2 = t_1$$

$$\Delta \bar{t} = \bar{t}_1 - \bar{t}_2 = t_2 - t_1 = -(t_1 - t_2) = -\Delta t$$

$$\Delta \bar{t} - \Delta t = -\Delta t - \Delta t = -2\Delta t = -\frac{2lv^2}{c^3}$$

Let T be the time period, n the frequency.

Then $n T = 1$ Phase difference $= \frac{\text{time difference}}{T}$

$$= -\frac{2lv^2}{c^3} \cdot \frac{n}{1} = -\frac{2nlv^2}{c^3}$$

Phase difference $= -\frac{2nlv^2}{c^3}$

Hence there could be expected a shift in fringes due to the above difference but no shift was observed. The experiment was repeated with multiple mirrors to increase the distance l but the result remains unchanged, i.e. all gave null results. One possible explanation could be v = 0 but this assumption is in contradiction with Fresnel's law of drift. Trouton and Noble in 1904 repeated the experiment using electromagnetic waves

instead of visible light but no shift in fringes was observed. All recent efforts, carefully avoiding possible errors could no bring any appreciable change in the original result.

6. Explanation of negative results:

(i) Drag theory:

As stated earlier, one possible explanation is v = 0, i.e., velocity of earth relative to ether is zero.

This gives $t_1 = t_2$. For:

$$t_1 = \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right) \quad , \quad t_2 = \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right)$$

Then there is no relative velocity between the earth and the ether. In other words, the ether is dragged with the motion of the earth with the same velocity as the earth. However if this explanation is accepted, there should be no aberration of light. Even if the ether is considered to be dragged partially, the absence of shift in fringes and the value of aberration cannot be explained simultaneously.

(ii) Lorentz and Fitzgerald contraction hypothesis:

Fitzgerald, in 1892, proposed a hypothesis which was elaborated by Lorentz and hence the name "Lorentz Fitzgerald contraction hypothesis" to explain negative results of Michelson-Morley experiment. Their hypothesis is "All

material bodies moving with velocity v are contracted in the direction of motion by a factor $(1 - \beta^2)^{1/2}$ where $\beta = v/c$".

According to this hypothesis if lo is the length of a body at rest relative to ether and l its length when it is in motion with velocity v relative to ether, then $l = l_o (1 - \beta^2)^{1/2}$.

By this idea they give an explanation for the negative results of Michelson-Morley experiment as follows:

$$t_1 = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right) \quad , \quad t_2 = \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right)$$

According to Lorentz and Fitzgerald, this becomes:

$$t_1 = \frac{2l}{c}\left(1 - \frac{v^2}{c^2}\right)^{1/2}\left(1 + \frac{v^2}{c^2}\right) = \frac{2l}{c}\left(1 - \frac{v^2}{2c^2}\right)\left(1 + \frac{v^2}{c^2}\right)$$

$$= \frac{2l}{c}\left(1 + \frac{v^2}{c^2} - \frac{v^2}{2c^2} - \frac{v^4}{2c^4}\right) = \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right) = t^2,$$

neglecting the term $\frac{v^4}{c^4}$.

$\therefore t_1 = t_2$ Therefore the times taken by reflected and transmitted beams are the same so that there is no phase difference between the two beams. Consequently no shift in fringes is observed.



Chapter 2

Lorentz Transformations

Introduction:

We enunciate the fundamental postulate of theory of relativity, "physical law is independent of the inertial frame of reference, i.e., a physical law has the same meaning in all inertial of frames." This postulate is called the postulate of covariance of physical law.

As an example we take the speed of light to be a physical law i.e., the speed of light is the same for all inertial frames. It does not depend on the its direction or the speed of the earth.

1. The new concept of space and time:

Lorentz and Fitzgerald explained the negative result of Michelson and Morley's experiment. These mathematicians changed the concept of space and rime considered under Newtonian mechanics. According to them, material body moving with city v through ether is contracted by the factor $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ in the direction of motion of body. The failure of Michelson and Morley's experiment led Einstein to formulate the new concept of space and time. He ruled out the ether

hypothesis. He said that no experiment could detect the velocity of earth through ether.

His conclusion was that the motion relative to material bodies has physical significance while the motion through ether is meaningless. In other words, there is no such thing as an absolute motion and all motions are relative. The physical laws are independent of the motion of observer.

X'———————•———————X
O ⟶

Fig. 1.

Suppose O is the middle point of the straight line X'X and v is the velocity of earth. Suppose that the two clocks giving absolute time are kept at X' and X. In any co-ordinate system, the events occurring at X' and X are said to be simultaneous, if the clocks placed at X' and X indicate the same time when the events occur.

It can not be so. For, if a light signal is given at X' and X simultaneously, then these light signals will not reach simultaneously the observer at O, on account of the fact that the time t_1 taken by the light signal from X' to O is $\frac{OX'}{c-v}$ and the time t_2 taken by the light signal from X to O is such that

$$t_1 = \frac{OX'}{c-v} \neq \frac{OX}{c+v} = t_2 \quad , \quad \text{as } c - v \neq c + v \quad , \quad OX' = OX$$

Therefore the time is not absolute, It varies from one internal system to another inertial system, i.e., $t' \neq t$. Therefore Einstein further ruled out the concept of absolute time.

According to Galiean transformations, $x' = x - vt$. Since time t is not absolute and therefore the distance between two points measured in two internal systems cannot be absolute, i.e., it cannot be expressed by classical equation $x' = x - vt$. Hence Einstein also ruled out the concepet of absolute space. He said that the theory of relativity having new concept of space and time is not valid for mechanical phenomenon and also for all optical and electromagnetic phenomenon. The theory of relativity is divided into two parts:

(i) Special theory of relativity.

It deals with inertial system, i.e., systems moving with uniform velocity.

(ii) General theory of relativity.

It deals with non-inertial system, i.e., systems moving with accelerated velocity.

2. Postulates of special theory of relativity:

(i) The natural laws must preserve their forms relative to alt observers in a state of relative uniform motion.

According to this postulate, velocity is not absolute but relative. It is a fact drawn from the failure of Michelson and Morley experiment, which was performed to determine velocity of earth through ether.

(ii) The velocity of light in vaccum is independent of the velocity of observer or the velocity of the source.

According to Galilean transformations, this postulate is not true. In fact, it is confirmed expertmentally that the velocity of light calculated by any method is constant. The second postulate is important in the sense that it gives a clear distinction between, classical theory and Einstein theory of relativity.

3. Lorentz transformation equations:

Consider two inertial frames of reference S and S' where S' is moving with uniform velocity v along X'-axis relative to S. Let the observers in the two systems be situated at the origins O and O' of and S' respectively. They are observing the same event at any point P "whose co-ordinates are {x , y , z , t} and {x' , y' , z' , t'} in S and S' respectively. The rectangular axes X' , Y' , Z' are parallel to X , Y , Z respectively. The choice of the origins of the two systems is such that their origins coincide at t = 0 , t' = 0.

It means that the axes X and X' are coincident permanently. Points which are at rest relative to S' will move the velocity v relative to S in X-direction.

In particular the point $x' = 0$ will move with velocity v in X-direction, i.e., $x' = 0$ will be identical with $x = vt$ so that:

$$x' = \alpha\ (x - vt) \qquad (1)$$

Where α is some function of v.

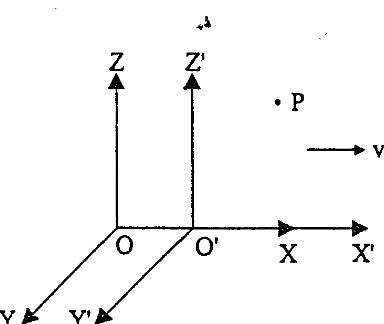


Fig. 2.

Since the velocity of S' is only along X-axis. Hence by symmetry,

$$\left.\begin{array}{l} y' = y \\ z' = z \end{array}\right\} \qquad (2)$$

To complete the set of equations, we have to formulate an equation in t and t'. t' depends on t , x , y , z linearly. For the reason of symmetry, we assume that^ t' does not depend upon y and z. otherwise clocks in S' would appear to disagree as observed from S. So we have:

$$t' = \beta\ t + \gamma\ x \qquad (3)$$

Where β and γ both are functions of v only. We are to determine the unknowns α , β , γ.

Let us assume that at time t = 0 a spherical wave of light signal leaves O which coincides with O' at that moment. Since the velocity of light in both system is the same and so the speed of propagation is the same in all direction and equal to c in terms of either set of co-ordinates. Its Progress is therefore described by either of the two equations.

$$x^2 + y^2 + z^2 = c^2t^2 \qquad (4)$$

$$x'^2 + y'^2 + z'^2 = c^2\, t'^2 \qquad (5)$$

Putting the values of x' , y' , z' in (5):

$$\alpha^2\,(x - v\,t)^2 + y^2 + z^2 = c^2(\beta\, t + x\,\gamma)^2$$

Or

$$x^2\,(\alpha^2 - c^2\,\gamma^2) + y^2 + z^2 - 2\,x\,t\,(\alpha^2 v + c^2\beta\gamma) = (c^2\beta^2 - \alpha^2 v^2)\, t^2$$

This equation and (4) represent the same motion. Hence, on comparing the coefficients of various terms, we obtain:

$$\alpha^2 - c^2\gamma^2 = 1 \qquad (6a)$$

$$\alpha^2 v + c^2\beta\,\gamma^2 = 0 \qquad (6b)$$

$$c^2\beta^2 - \alpha^2 v^2 = c^2 \qquad (6c)$$

$$v \times (6a) - (6b) \text{ gives } - c^2\gamma^2 v - c^2\beta\,\gamma = v$$

Or

$$v\,(1 - c^2\gamma^2) + c^2\beta\,\gamma = 0 \qquad (7)$$

Similarly $\quad v^2 \times (6a) + (6c) \text{ gives } - v^2c^2\gamma^2 + c^2\beta^2 = v^2 + c^2$

Or $\quad -v^2(1+c^2\gamma^2)+c^2\beta^2=c^2 \qquad (8)$

Now $\quad v\times(7)+(8)$ gives $vc^2\beta\gamma+c^2\beta^2=c^2$

Or $\quad v\,\beta\gamma+\beta^2=1$

Or $\quad \beta^2-1=v\,\beta\gamma \qquad (9)$

Eliminating γ between (7) and (9):

$$v\left\{1+c^2\left(\frac{\beta^2-1}{v\beta}\right)^2\right\}+c^2\left(\frac{1-\beta^2}{v}\right)=0$$

Or $$\frac{v\,[v^2\beta^2+c^2(\beta^2-1)^2]}{v^2\beta^2}+\frac{c^2(1-\beta^2)}{v}=0$$

Or $\quad v^2\beta^2+c^2(\beta^2-1)^2+c^2\beta^2(1-\beta^2)=0$

Or $\quad \beta^2[v^2+c^2-2c^2]+c^2=0$

Or $$\beta^2=\frac{c^2}{c^2-v^2}$$

Using this in (6c).

$$\frac{c^4}{c^2-v^2}-\alpha^2v^2=c^2 \quad \text{or} \quad \alpha^2v^2=\frac{c^4}{c^2-v^2}-c^2=\frac{v^2c^2}{c^2-v^2}$$

$$\alpha=\beta=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Using this in (6b).

$$\gamma = \frac{-\alpha^2 v}{c^2 \beta} = \frac{-\alpha v}{c^2} = \frac{-\beta v}{c^2}$$

Also $$\alpha = \beta = \frac{1}{\sqrt{(1 - v^2 / c^2)}}$$

$$x' = \alpha(x - vt) = \frac{1}{\sqrt{(1 - v^2 / c^2)}}(x - vt)$$

$$t' = \beta\, t + \gamma\, t - \beta\, t - \beta \frac{vx}{c^2} = \beta \left(t - \frac{vx}{c^2}\right)$$

Thus, Lorentz transformation equations are:

$$x' = \beta\, (x - vt) \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = \beta \left(t - \frac{vx}{c^2}\right)$$

Where $$\beta = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}.$$

Note 1: If v is very small, then $\frac{v}{c} \to 0$ so that $\beta \to 1$. In this case Lorentz transformation equations become:

$$x' = (x - vt) \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = t$$

These are Galilean transformation. Thus Lorentz transformation reduce to Galilean transformation if $v << c$.

Note 2: Solving Lorentz transformation for x , y , z :

$$x = \frac{x'}{\beta} + v\left(\frac{t'}{\beta} + \frac{vx}{c^2}\right) = \frac{1}{\beta}(x'+vt') + \frac{v^2x}{c^2}$$

Or $$\left(1 - \frac{v^2}{c^2}\right) x = \frac{1}{\beta}(x'+vt')$$

Or $$\frac{x}{\beta^2} = \frac{1}{\beta}(x + vt')$$

Or $$x = \beta\ (x'+vt')$$

$$t' = \beta\left(t - \frac{vx}{c^2}\right) \text{ gives:}$$

$$t = \frac{t'}{\beta} + \frac{v}{c^2}\beta\ (x'+vt') = \frac{t'}{\beta}\left(1 + \frac{v^2}{c^2}\beta^2\right) + \frac{vx'}{c^2}\beta$$

$$= \frac{t'}{\beta} + \left(\frac{1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}\right) + \frac{vx'}{c^2}\beta = \frac{t'}{\beta\left(1 - \frac{v^2}{c^2}\right)} + \frac{vx'}{c^2}\beta$$

$$= \frac{t'}{\beta}\beta^2 + \frac{vx'}{c^2}\beta = \beta\left(t' + \frac{vx'}{c^2}\right)$$

Thus $$x = \beta\ (x'+vt')\ ,\quad y = y'\ ,\quad z = z'\ ,\quad t = \beta\left(t' + \frac{vx'}{c^2}\right)$$

These are called Lorentz inverse transformations. Comparing these equations with Lorentz transformations, we see that the system S is moving with velocity $-v$ relative to S' along X-axis.

4. Consequences of Lorentz transformations:

Lorentz and Fitzgerald contraction:

Lorentz transformations for a system S' moving with velocity v along X-axis relative to a system S are given by:

$$x' = \beta\ (x - vt) \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = \beta\ (t - vx/c^2)$$

$$\beta = \frac{1}{\sqrt{(1 - v^2/c^2)}}$$

Let there be a rod of length l, which is placed parallel to Y-axis It y_1 , y_2 refer to end points of the rod, then:

$$l = y_2 - y_1$$

But $y'_2 - y'_1 = y_2 - y_1 = l$ i.e. $y'_2 - y'_1 = l$

This proves that the lengthy of the rod remains the same, when it is measured by an observer in S'. Consequently the lengths perpendicular to the direction of motion are unchanged.

Next we suppose that a rod is placed along Z-axis. Let the length of the rod in S system be l. Then we can take:

$$l = x_2 - x_1$$

where x_1 and x_2 refer to end points of the rod. The measurements of both ends are taken at the same t in S so that:

$$t_2 - t_1 = 0$$

Let x'_1 and x'_2 be the end points of the same rod along X-axis at the time t', measured by an observer S' system. The observations of both end points of the rod are taken at the same time t' so that:

$$t'_1 = t'_2 = t'$$

l' = length of rod in S' system $= x'_2 - x'_1$

$$x_1 = \beta\ (x'_1 + vt'_1) \quad , \quad x_2 = \beta\ (x'_2 + vt'_2)$$

$$l = x_2 - x_1 = \beta\ (x'_2 - x'_1) + v\ \beta\ (t'_2 - t'_1) = \beta\ l' \qquad \text{For } t'_1 = t'_2$$

$$\therefore l = \beta\ l' \quad \text{or} \quad l' = \frac{1}{\beta} = l\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$l' = l\left(1 - \frac{v^2}{c^2}\right)^{1/2} < l \qquad \therefore l' < l$$

This proves that the apparent length of a rigid body in the direction of its motion is reduced by the factor $\left(1 - \frac{v^2}{c^2}\right)^{1/2}$.

Hence: Every rigid body appears to be longest when at rest relative to the observer {i.e. $l > l'$}. If the body moves with uniform velocity v relative to the observer, then its apparent length is contracted by the factor $\left(1 - \frac{v^2}{c^2}\right)^{1/2}$ in the direction of relative motion $l' < l$.

5. Time dilation:

Consider two frames of references S and S', S is moving with uniform velocity v along X-axis.

Let a clock be placed in the system S at a point $x = x_1$. Let this clock give a signal at time $t = t_1$ in S and let t'_1 be the time measured by an observer in S', corresponding to it. Then, by Lorentz transformations:

$$t'_1 = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \qquad (1)$$

Where $$\beta = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Let the clock give another signal at time t_2 in S and the corresponding time in S' is t'_2. Then:

$$t'_2 = \beta\left(t_2 - \frac{vx_1}{c^2}\right) \qquad (2)$$

Write $\Delta t = t_2 - t_1 \quad , \quad \Delta t' = t'_2 - t'_1$

Then (2) , (1) gives:

$$\Delta t' = \beta\ \Delta t \qquad (3)$$

or $$\Delta t' = \Delta t\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \left(1 + \frac{v^2}{2c^2}\right)\Delta t > \Delta t$$

$\therefore \Delta t' > \Delta t$

The physical significance of equation (3) is as follows:

The interval $\Delta t'$, as it appears to the observer in motion, is lengthened, i.e. the time is dilated and hence the name "time dilation." It means that the time interval Δt appears to be dilated or lengthened by the factor β to the moving observer. Thus, according to (3), a clock moving relative to an observer is found to run more slowly than the one, which is at rest relative to him. In other words, a physical process of finite duration will take a much longer time in a moving frame than it would be in a frame at rest.

Reversing our process, let us suppose that the clock is placed at a point x'_1 in the system S'. If the clock gives a signal at time t'_1 in S' and t_1, is the corresponding time in S, measured by the observer in S. Then, by Lorentz inverse transformations:

$$t_1 = \beta \left(t'_1 + \frac{vx'_1}{c^2} \right)$$

Similarly, $$t_2 = \beta \left(t'_2 + \frac{vx'_1}{c^2} \right)$$

Then $$\Delta t = t_2 - t_1 = \beta \; (t'_2 - t'_1) = \beta \; \Delta t'$$

$$\Delta t = \beta \; \Delta t' \qquad (4)$$

This $\Delta t > \Delta t'$

It says that the time interval $\Delta t'$ appears to be dilated by the factor β to an observer moving with velocity -v relative to S'.

Hence from above reasoning we may say, "A moving clock always appears to go slow". Consequently to the observer in motion the clock at Test appears to be retarded by the factor $\sqrt{\left(1-\frac{v^2}{c^2}\right)}$. This is apparent retardation of clocks. From what has been done it follows that:

Every clock appears to go at its fastest rate when it is at rest relative to the observer. If the clock moves w . r . t. the observer with velocity v, then it appears to go at its slowest rate by the factor $\sqrt{\left(1-\frac{v^2}{c^2}\right)}$.

Deduction:

According to the observer in S, the clock in S' is going slow but from the point of view of S' it is S's clock, which is moving fast, and therefore when he comes back to S's he found the just reversed phenomenon.

An interesting example of time dilation:

Imagine that once a 40 years old scientist falls in love with years old girl who is his laboratory assistant. They want to

marry; but they feel that their marriage cannot be welcomed by the society due to the age difference. Scientist plans to marry using the principle of time dilation of relativity. So he synchronizes his clock with that of his assistant and goes to a long journey in a rocket with velocity 0.999c. He returns back when his clock reads one year. But he finds in her clock 22.7 years have:

$$\text{For } \Delta t' = \frac{\Delta t}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} = \frac{1}{\sqrt{\left[1-\left(\frac{999c}{c}\right)^2\right]}} = 22.7 \text{ years}$$

passed. Now scientist is 40 + 1 = 41 years old and his assistant is 16 + 22.7 = 38.7 years old. Their age difference barrier has now been overcome. Therefore they marry.

6. Simultaneity:

Any two events are said to be simultaneous if they occur at the same time.

Consider two frames of reference S and S' , S' is moving with velocity v along X-axis. Also let two events occur simultaneously in S at two different points P_1 (x_1 , y_1 , z_1 , t_1) and P_2 (x_2 , y_2 , z_2 , t_2) so that:

$$x_1 \neq x_2 \quad , \quad t_1 = t_2$$

The events are simultaneous in S $\Rightarrow t_1 = t_2$. Let t'_1 and t'2 be time in S' corresponding to the times t_1 and t_2 in S. By Lorentz transformations:

$$t'_1 = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \quad , \quad t'_2 = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

Hence $$t'_2 - t'_1 = \beta\ (t_2 - t_1) + \beta\ \frac{v}{c^2}(x_1 - x_2)$$

$$= \beta\ \frac{v}{c^2}(x_1 - x_2) \quad \text{For } t_1 = t_2$$

But $x_1 \neq x_2$ so that the last says that $t'_1 \neq t'_2$. This means that the same two events are not simultaneous in S'.

Two events at different places P_1 and P_2 which are simultaneous for an observer at rest in S, are no longer simultaneous to an observer S' which is moving with velocity v relative to S along X-axis. It shows that simultaneity is not absolute, but it is relative.

Problem: Prove that "simultaneity" has only a relative and absolute meaning.

7. Relativistic formulae for composition of velocities:

Consider two frames of reference S and S' , S' is moving with velocity v along X-axis relative to S. Let (x , y , z , t) and (x', y', z', t') be co-ordinates of a moving point P is S and S'

respectively. The velocity components of P in S' and S' are given by (u_x, u_y, u_z) and (u'_x, u'_y, u'_z), where:

$$u_x = \frac{dx}{dt}\ ,\ u_y = \frac{dy}{dt}\ ,\ u_z = \frac{dz}{dt}$$

$$u'_x = \frac{dx'}{dt'}\ ,\ u'_y = \frac{dy'}{dt'}\ ,\ u'_z = \frac{dz'}{dt'}$$

Lorentz reverse transformations are

$$x = \beta\ (x'+vt')\ ,\ y = y', z = z'\ \ t = \beta\left(t'+\frac{vx'}{c^2}\right)$$

Where $$\beta = 1\Big/\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

Taking differentials:

$$dx = \beta\ (dx' + vdt')\ ,\ dy = dy'\ ,\ dz = dz'\ ,\ dt = \beta\left(dt'+\frac{v}{c^2}\ dx'\right)$$

$$u_x = \frac{dx}{dt} = \frac{\beta\ (dx'+vdt')}{\beta\left(dt'+\frac{v}{c^2}dx'\right)} = \left(\frac{dx'}{dt'}+v\right)\Big/\left(1+\frac{v}{c^2}\frac{dx'}{dt'}\right)$$

$$u_x = \frac{u_x'+v}{1+\frac{v}{c^2}u_x'}$$

$$u_x = \frac{dy}{dt} = \frac{dy'}{\beta\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{dy'/dt'}{\beta\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} = u_y'/\beta\left(1 + \frac{v}{c^2}u_x'\right)$$

Similarly $u_z = u_z'/\beta\left(1 + \frac{v}{c^2}u_x'\right)$

Then
$$u_x = \frac{u_x' + v}{\left(1 + \frac{v}{c^2}u_x'\right)}$$

$$u_y = \frac{u_y'}{\beta\left(1 + \frac{v}{c^2}u_x'\right)}$$

$$u_z = \frac{u_z'}{\beta\left(1 + \frac{v}{c^2}u_x'\right)}$$

The velocity components (u_x , u_y , u_z) represent the result of compounding the velocity (u_x' , u_y' , u_z') and (v , 0 , 0).

Results:

(I) If we write $u_x' = u'$, $u_y' = 0$, $u_z' = 0$ then:

$$u_x = (u' + v)\Big/\left(1 + \frac{u'v}{c^2}\right)$$

This is the relativistic law of addition of two velocities in the same direction i.e. X-axis.

(2) The velocity of light is an absolute constant independent of the motion of the reference system.

For proving this, take $u' = c$,

Then $$u_x = \frac{c+v}{1+\frac{cv}{c^2}} = \frac{c\,(c+v)}{c+v} = c$$

$\therefore u_x = c$ Hence the result follows.

The result (2) is also expressible as: the velocity of light c cannot be changed by adding to it a velocity smaller than c.

(3) The resultant of two velocities each of which is less than c is also less than c.

Proof: Let $u' = c - l$, $v = c - w$, where $l, m > 0$. l, m being constants.

$$u_x = \frac{u'+v}{1+\frac{u'v}{c^2}} = \frac{c-l+c-m}{1+\frac{(c-l)(c-m)}{c^2}} = \frac{2c-l-m}{1+\frac{c^2-(l+m)\,c+lm}{c^2}}$$

$$= c\left(\frac{2c-l-m}{\frac{2c^2-(l+m)\,c+lm}{c}}\right) = c\left(\frac{2c-(l+m)}{2c-(l+m)+\frac{lm}{c}}\right)$$

$$l, m > 0 \Rightarrow \frac{lm}{c} > 0$$

$$\Rightarrow 2c-(l+m)+\frac{lm}{c} > 2c-(l+m)$$

$$\Rightarrow \frac{2c-(l+m)}{2c-(l+m)+\frac{lm}{c}} < 1$$

$$\Rightarrow u_x < c$$

(4) The addition of any velocity to the velocity of light is simply equal to c.

Proof: Let $u' = c$, $v = c$.

Then $$u_x = \frac{u+v}{1+\frac{u'v}{c^2}} = \frac{c+c}{1+\frac{cc}{c^2}} = \frac{2c}{2} = c$$

This result proves that the velocity of light is an upper limit of the velocity of a particle with which a particle can move in nature.

This fact is also expressed by saying that it is impossible to send out signals with a velocity greater than that of light.

(5) The velocity of light is called fundamental velocity due to the properties listed below.

(1) It is constant in all directions.

(2) It is the same for all observers independent of the velocities of the source and the observer.

(3) It is invariant for any two systems joined by Lorentz transformations.

(4) The addition of any velocity to the velocity of light is simply equal to c.

(5) It is impossible to send out signals with a velocity greater then c.

8. Relativistic formulae for composition of accelerations:

First we shall find expression for u_x , u_y , u_z as Article 7.

We have $t = \beta\left(t' + \frac{vx'}{c^2}\right)$ so that $dt = \beta\left(dt' + \frac{v}{c^2}dx'\right)$

$$\frac{dt}{dt'} = \beta\left(1 + \frac{v}{c^2}u_x'\right)$$

$$a_x = \frac{du_x}{dt} = \frac{d}{dt}\left(\frac{u_x' + v}{1 + \frac{u_x' v}{c^2}}\right) = \frac{d}{dt'}\left(\frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}\right) \cdot \frac{dt'}{dt}$$

$$= \frac{a_x'\left(1 + \frac{vu_x'}{c^2}\right) - \frac{v}{c^2}a_x'(v + u_x')}{\left(1 + \frac{vu_x'}{c^2}\right)^2} \cdot \frac{1}{\beta\left(1 + \frac{vu_x'}{c^2}\right)}$$

$$a_x = \frac{a_x'\left(1 - \frac{v^2}{c^2}\right)}{\beta\left(1 + \frac{vu_x'}{c^2}\right)^3} = \frac{a_x'}{\left(1 + \frac{v}{c^2}u_x'\right)^3 \beta^3}$$

$$a_y = \frac{du_y}{dt} = \frac{du_y}{dt'} \cdot \frac{dt'}{dt} = \frac{d}{dt'}\left[\frac{u_y'}{\beta\left(1+\frac{vu_x'}{c^2}\right)}\right] \cdot \frac{dt'}{dt}$$

$$= \frac{a_y'\left(1+\frac{v}{c^2}u_x'\right) - \frac{v}{c^2}a_x'\ u_y'}{\beta\left(1+\frac{vu_x'}{c^2}\right)^2} \cdot \frac{1}{\beta\left(1+\frac{v}{c^2}u_x'\right)}$$

$$a_y = \frac{1}{\beta^2}\left[\frac{a_y'}{\left(1+\frac{vu_x'}{c^2}\right)^2} - \frac{\frac{v}{c^2}a_x'\ u_y'}{\left(1+\frac{v}{c^2}u_x'\right)^3}\right]$$

Then $$a_x = \frac{a_x'}{\beta^3 M^3} \quad , \quad a_y = \frac{1}{\beta^2}\left[\frac{a_y'}{M^2} - \frac{\frac{v}{c^2}a_x'\ u_y'}{M^3}\right]$$

Similarly $$a_z = \frac{1}{\beta^2}\left[\frac{a_z'}{M^2} - \frac{\frac{v}{c^2}a_x'\ u_z'}{M^3}\right]$$

Where $$M = 1 + \frac{vu_x'}{c^2}$$

From these equations, it is clear that the components of acceleration in S' are constants, but the component of

acceleration in S system are not constants, in general due to the fact that they contain components of velocity also.

Deduction: Now we consider a Particular case in which the particle under consideration is at rest relative to S' so that:

$$u_x' = 0 \quad , \quad u_y' = 0 \quad , \quad u_z' = 0$$

Consequently $M = 1 + (vu_x' / c^2) = 1$ or $M = 1$

Now the foregoing expressions take the form:

$$\dot{u}_x = \frac{\dot{u}_x'}{\beta^3} \; , \; \dot{u}_y = \frac{\dot{u}_y'}{\beta^2} \; , \; \dot{u}_z = \frac{\dot{u}_z'}{\beta^2}$$

Taking $\dot{u}_x = f_x \; , \; \dot{u}_y = f_y \; , \; \dot{u}_z = f_z$ we get:

$$f_x = \left(1 - \frac{v^2}{c^2}\right)^{3/2} f_x', \; f_y = \left(1 - \frac{v^2}{c^2}\right) f_y', \; f_z = \left(1 - \frac{v^2}{c^2}\right)^{3/2} f_z'$$

9. Relativity of time. Proper time:

The time recorded by a clock moving with a given system is called proper time for that system.

Consider two frames of reference S and S', S' is moving with velocity v relative to S along X-axis. Now we imagine that the observer in S is at rest and the observer in S' is in an aeroplane who is moving with velocity v along X-axis.

Suppose the observer in S records the time t for the journey of the observer in the aeroplane. Let: t' be the time recorded by

the observer in the aeroplane. Let t' be the time recorded by the observer in S', corresponding to time t. Then:

$$x = vt \qquad \text{(distance = velocity × time)}$$

$$t' = \beta\left(t - \frac{vx}{c^2}\right) \qquad \text{by Lorentz transformations.}$$

$$= \beta\left(t - \frac{vvt}{c^2}\right) = \beta t\left(1 - \frac{v^2}{c^2}\right) = \beta t\frac{1}{\beta^2} = \frac{t}{\beta}$$

$$= t\left(1 - \frac{v^2}{c^2}\right)^{1/2} \quad , \qquad \beta = 1\Big/\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

$$< t$$

$$\therefore \; t' < t$$

i.e., time recorded in S' < time recorded in S. From this it follows that the two clocks in the two systems run at different rates. Also it is clear that a 'moving clock runs more slowly than a stationary one. We thus conclude that there are two proper times.

Theorem: The results of two successive Lorentz transformation is a Lorentz transformation.

Proof: To prove the theorem, we have to prove that the resultant of two successive Lorentz transformations is a Lorentz transformation.

Consider three frames of reference S , S' , S'', where S' is moving with velocity v relative to S along X-axis and S'' is moving with velocity v' relative of S' along X-axis. By Lorentz transformation equations, the frames S and S' can be related as:

$$x' = \beta\ (x - vt)\quad ,\quad y' = y\quad ,\quad z' = z\quad ,\quad t' = \beta\left(t - \frac{vx}{c^2}\right)$$

Where $$\beta = 1 \Big/ \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \qquad (1)$$

Similarly, S' and s'' can be related as:

$$x'' = \beta'\ (x' - v't')\quad ,\quad y'' = y'\quad ,\quad z'' = z',\quad t'' = \beta'\left(t - \frac{v'x'}{c^2}\right)$$

Where $$\beta' = 1 \Big/ \sqrt{\left(1 - \frac{v'^2}{c^2}\right)} \qquad (2)$$

Let v'' be the resultant of v and v' so that $v'' = \dfrac{v + v'}{1 + \dfrac{vv'}{c^2}}$ (3)

v'' is the velocity of the system S'' relative to S.

Write $$\beta'' = 1 \Big/ \sqrt{\left(1 - \frac{v''x}{c^2}\right)}$$

If we show that:

$$x''=\beta''\ (x-v''t)\ ,\quad y''=y\ ,\quad z''=z\ ,\quad t''=\beta''\left(t-\frac{v''x}{c^2}\right)(4)$$

$$\frac{1}{\beta''^2}=1-\frac{v''^2}{c^2}=1-\frac{1}{c^2}\left(\frac{v+v'}{1+\frac{vv'}{c^2}}\right)^2=\frac{c^2\left(1+\frac{vv'}{c^2}\right)-(v+v')^2}{c^2\left(1+\frac{vv'}{c^2}\right)^2}$$

$$=c^2+2vv'+\frac{v^2v'^2}{c^2}-(v^2+v'^2+2vv')/c^2\left(1+\frac{vv'}{c^2}\right)^2$$

$$=c^2\left[1+\frac{v^2v'^2}{c^4}-\frac{v^2}{c^2}-\frac{v'^2}{c^2}\right]\Big/c^2\left(1+\frac{vv'}{c^2}\right)^2$$

$$=\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{v'^2}{c^2}\right)\Big/\left(1+\frac{vv'}{c^2}\right)^2$$

or $$\beta''=\left(1+\frac{vv'}{c^2}\right)\Big/\sqrt{\left(1-\frac{v^2}{c^2}\right)}\ \sqrt{\left(1-\frac{v'^2}{c^2}\right)}=\beta\beta'\left(1+\frac{vv'}{c^2}\right)$$

Thus $$\beta''=\beta\beta'\left(1+\frac{vv'}{v'}\right) \qquad (5)$$

Writing (2) with the help of (1):

$$x''=\beta'(x'-v't')=\beta'\left[\beta\ (x-vt)-v'\beta\left(t-\frac{vx}{c^2}\right)\right]$$

$$= \beta'\beta \left[x \left(1 + \frac{vv'}{c^2}\right) - t\ (v + v') \right]$$

$$= \beta'\beta \left(1 + \frac{vv'}{c^2}\right) \left[x - \frac{v + v'}{2 + \frac{vv'}{c^2}} \cdot t \right] = \beta''(x - v''t) \quad \text{by (4)}$$

$$x'' = \beta''(x - v''t)$$

Again from (2) and (1):

$$t'' = \beta'\left(t' - \frac{v'x'}{c^2}\right) = \beta'\left[\beta \left(t - \frac{vx}{c^2}\right) - \frac{v'}{c^2}\beta\ (x - vt)\right]$$

$$= \beta'\beta \left[t \left(1 + \frac{vv'}{c^2}\right) - \frac{x}{c^2}\ (v + v') \right]$$

$$= \beta'\beta \left(1 + \frac{vv'}{c^2}\right) \left[t - \frac{v + v'}{1 + \frac{vv'}{c^2}} \cdot \frac{x}{c^2} \right]$$

$$= \beta'' \left(t - \frac{xv''}{c^2}\right) \quad \text{by (3) and (5):}$$

$$y'' = y' \quad , \quad y' = y \Rightarrow y'' = y$$

$$z'' = z' \quad , \quad z' = z \Rightarrow z'' = z$$

Thus we have shown that:

$$x'' = \beta''\ (x - v''t)\ , \quad y'' = y\ , \quad z'' = z\ , \quad t'' = \beta'' \left(t - \frac{v''x}{c^2}\right)$$

Solved problems

Problem 1- At what speed should a clock be moved so that it may appear to lose 1 minute in each hour.

Solution: Since the clock is to loose one minute in one hour. Hence the moving clock records 59 minutes for each hour recorded by clock stationary relative to the observer. Let v be the required speed of the moving clock. Given $\Delta t' = 59$ minutes, $\Delta t = 60$ minutes.

Evidently $\Delta t > \Delta t'$ Hence $\Delta t = \beta \Delta t' = \dfrac{\Delta t'}{\left(1-\dfrac{v^2}{c^2}\right)^{1/2}}$

or $\quad \left[1-\dfrac{v^2}{c^2}\right]^{1/2} \Delta t = \Delta t' \quad$ or $\quad 1-\left(\dfrac{v}{c}\right)^2 = \left(\dfrac{59}{60}\right)^2$

or $\quad 1-\left(\dfrac{59}{60}\right)^2 = \left(\dfrac{v}{c}\right)^2 \quad$ or $\quad v = c\,\dfrac{(1\times 119)^{1/2}}{60} = \dfrac{10.909}{60}c$

or $\quad v = 0.18\ c = 5.4\times 10^{10}\ \text{cm/sec}.$

Problem 2- A body has the dimensions represented by 6i + 7j meters in reference system S. How these dimensions will be represented in the system S' if S' is moving with velocity .6c along + ive x-axis i , j being unit vectors along respective axes.

Solution: By the result of Lorentz contraction:

$$l' = l\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

Given $v = .6c$ so that $1 - (v^2/c^2) = 1 - .36 = .64 = (.8)^2$

Hence $l' = 6\ (.8) = 4.8$

The dimension of the body along x-axis in S' system is 4.8. But there is no contraction in the direction of y-axis, since there is no motion along y-axis in S'. Hence the body will be represented by $4.8i + 7j$ meters in S' system. **Ans.**

Problem 3- The lenght of a rocket ship is 100 meters on the ground. When it is in flight its length observed on the ground is 99 meters, calculate its speed.

Solution: By the result of Lorentz contraction:

$$l' = l\sqrt{[1 - (v^2/c^2)]}$$

Hence $\qquad 99 = 100\ [1 - (v^2/c^2)]^{1/2}$ as $l' < l$

Or $\quad \left(\frac{99}{100}\right)^2 = 1 - \frac{v^2}{c^2}$ or $\frac{v^2}{c^2} = \frac{199}{10^4}$ or $\frac{v}{c} = \frac{\sqrt{(199)}}{100}$

$$v = \frac{(199)^{1/2}}{100} \times 3 \times 10^8 \text{ m/sec.}$$

$$= 3 \times 10^6 (199)^{1/2} = 42.3 \times 10^6 \text{ m/sec}$$

Problem 4- A rod has length 100 cm. when the rod is in a satellite moving with velocity 0.8c relative to laboratory what is length of the rod as determined by an observer, (a) in the satellite and (b) in the laboratory.

Solution: (a) The observer in the satellite is at rest relative to the rod as the rod is in the satellite. So the length of the rod relative to this observer is 100 cm.

(b) Now the observer is in the laboratory. So the rod is in motion w . r . t. the observer.

$$\because l' = l\sqrt{[1-(v^2/c^2)]}$$

$$\therefore l' = 100\ [1-(.8)^2]^{1/2} = 100 \times .6 = 60\ \text{cm}$$

(a) 100 cm , (b) 60 cm

For l = length of rod when it is at rest w . r . t. the observer.

l' = length of rod when it is in motion w . r . t. observer.

Problem 5- Calculate the length of a rod moving with a velocity of 0.8c in a direction inclined at 60° to its own length. Proper length of the rod is given to be 100 cm.

Solution: Suppose the rod is moving with velocity $v = .8c$ along x-axis and the rod is inclined at 60° with x-axis.

Contraction will take place only in x-direction and not in y-direction. Hence $l_x = l \cos 60°$, $l_y = l \sin 60°$.

$$\bar{l} = il_x + jl_y$$

$$l_x' = l_x\sqrt{[1-(v^2/c^2)]} \quad , \quad l_y' = l_y = l\sin 60° = (\sqrt{3/2})\ l$$

$$l_x' = (l\cos 60°)\sqrt{[1-(.64)]} = l(1/2)(.6) = .3l$$

$$l' = [(l_x')^2 + (l_y')^2]^{2/1} = l\ [.09 + (3/4)]^{1/2}$$

$$= .916\ l = .916 \times 100 = 91.6$$

l = length of rod at rest , l' = length of moving rod.

The length of rod in motion is 91.6 cm.

Problem related to composition of velocities:

Problem 1- A particle instantaneously at rest in frame of reference S' experiences an acceleration in it represented by a vector:

$$f' = 3i + 4j + 12k$$

What is the acceleration measured from the frame of reference S; given S' move with velocity .98c relative to frame S along positive x-axis.

Solution: Given $f' = 3i + 4j + 12k = f'_x I + f'_y j + f'_z k$

Then $f'_x = 3$, $f'_y = 4$, $f'_z = 12$

We want to find:

$$f = f_x i + f_y j + f_z k$$

We know that:

$$f_x = (1-\beta^2)^{3/2} f'_x \ , \ f_y = (1-\beta^2)\ f'_y \ , \ f_z = (1-\beta^2)\ f'_z$$

(Refer Deduction of Article B)

Here $\quad \beta = (v/c) = .98$ and so $\quad 1-\beta^2 = 1-(.98)^2 = .04$

$$\therefore (1-\beta^2)^{3/2} = (.04)^{3/2} = (.2)^3 = .008$$

$$f_x = (1-\beta^2)^{3/2}\ f_x' = .008\times 3 = .024$$

$$f_y = (1-\beta^2)\ f_y' = .04\times 4 = .16$$

$$f_z = (1-\beta^2)\ f_z' = .04\times 12 = .48$$

$$f = if_x + jf_y + kf_z = .024i + .16j + .48k$$

Problem 2- If a photon travels the path in such a way that it r moves in x'y'-plane and makes an angle Ø with x-axis of system S', then $u_x^2 + u_y^2 = c^2$ for the system S.

Solution: By assumption, $u_x' = c\cos Ø$, $u_y' = c\sin Ø$ so that $(u_x')^2 + (u_y')^2 = c^2$.

To prove $u_x^2 + u_y^2 = c^2$.

Let the system S' be moving with velocity v relative to S. Then we have the relations:

$$u_x = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}} \quad , \quad u_y = \left(\frac{u_y'}{1 + \frac{vu_x'}{c^2}}\right) \cdot \frac{1}{\beta}$$

Where $$\beta = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

$$u_x^2 + u_y^2 = \left[\frac{c\cos\phi + v}{1+\frac{v}{c}\cos\phi}\right]^2 + \left[\frac{c\sin\phi}{1+\frac{v}{c}\cos\phi}\right]^2 \left(1-\frac{v^2}{c^2}\right)$$

$$= \frac{c^2}{(c+v\cos\phi)^2}[c^2\cos^2\phi + v^2 + 2vc\cos\phi + \sin^2\phi\ (c^2 - v^2)]$$

$$= \frac{c^2}{(c+v\cos\phi)^2}[c^2 + v^2 + 2vc\cos\phi - v^2\sin^2\phi]$$

But $$v^2 - v^2\sin^2\phi = v^2\cos^2\phi$$

$$= \frac{c^2}{(c+v\cos\phi)^2}(c+v\cos\phi)^2 = c^2$$

Problem 3- If u and v are two velocities in the same direction and V their resultant velocity given by:

$$\tanh^{-1}\frac{V}{c} = \tanh^{-1}\frac{u}{c} + \tanh^{-1}\frac{v}{c}$$

Then deduce the law of composition of velocities from this equation.

Solution: Given that:

$$\tanh^{-1}\frac{V}{c} = \tanh^{-1}\frac{u}{c} + \tanh^{-1}\frac{v}{c}$$

This equation is expressible as:

$$\tfrac{1}{2}\log\frac{c+V}{c-V} = \tfrac{1}{2}\log\frac{c+u}{c-u} + \tfrac{1}{2}\log\frac{c+v}{c-v}$$

Or $$\log\frac{c+V}{c-V} = \log\frac{c+u}{c-u}\cdot\frac{c+v}{c-v}$$

This $$\Rightarrow \frac{c+V}{c-V} = \frac{c+u}{c-u}\cdot\frac{c+v}{c-v} = \frac{c^2+(u+v)c+uv}{c^2-(u+v)c+uv}$$

$$\Rightarrow \frac{c+V}{c-V} - 1 = \frac{c^2+(u+v)c+uv}{c^2-(u+v)c+uv} - 1$$

$$\Rightarrow \frac{2V}{c-V} = \frac{2(u+v)c}{c^2-(u+v)c+uv}$$

$$\Rightarrow \frac{c-V}{V} = \frac{c^2-(u+v)c+uv}{(u+v)c}$$

$$\Rightarrow \frac{c}{V} - 1 = \frac{c}{u-v} - 1 + \frac{uv}{c(u+v)}$$

$$\Rightarrow \frac{c}{V} = \frac{c}{u+v} + \frac{uv}{c(u+v)} = \frac{c^2+uv}{c(u+v)}$$

$$\Rightarrow \frac{c}{V} = \frac{c(u+v)}{c^2+uv} \Rightarrow V = \frac{c^2+uv}{c(u+v)}$$

$$\Rightarrow V = \frac{u+v}{1+\frac{uv}{c^2}}$$

This is the required expression for V.

Problem 4- Prove that four dimensional volume element dx , dy , dz , dt is invariant under Lorentz transformations.

Solutions: By Lorentz-Fitzerald contraction, we have:

$$dx' = \left(1 - \frac{v^2}{c^2}\right)^{1/2} dx$$

Also $\quad dy' = dy \quad , \quad dz = dz'$

By time dilation:

$$dt' = \frac{dt}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

$$dx' . dy' . dz' . dt' . = dx . \sqrt{\left(1 - \frac{v^2}{c^2}\right)} . dy . dz . dt \Big/ \sqrt{\left(1 - \frac{v^2}{c^2}\right)}$$

$$= dx . dy . dz . dt$$

$$dx' . dy' . dz' . dt' = dx . dy . dz . dt$$

This proves the required result.

Problem 5- Show that $x^2 + y^2 + z^2 - c^2t^2$ is Lorentz invariant.

Solution: By Lorentz transformation equations:

$$x' = \beta\ (x - vt) \quad , \quad y' = y \quad , \quad z' = z \ ,$$

$$t' = \beta\left(t - \frac{vx}{c^2}\right) \quad , \quad \beta = 1 \Big/ \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$x'^2 + y'^2 + z'^2 - c'^2t'^2$$

$$= \beta^2(x - vt)^2 + y^2 + z^2 - c^2\beta^2\left(t - \frac{vx}{c^2}\right)$$

$$= \beta^2\left[x^2 + v^2t^2 - 2vxt - c^2\left(t^2 + \frac{v^2x^2}{c^4} - \frac{2vtx}{c^2}\right)\right] + y^2 + z^2$$

$$= \beta^2\left(1 - \frac{v^2}{c^2}\right)x^2 - c^2t^2\left(1 - \frac{v^2}{c^2}\right)\beta^2 + y^2 + z^2$$

$$= x^2 - c^2t^2 + y^2 + z^2 \quad \text{For} \quad \beta^2 = 1\Big/\left(1 - \frac{v^2}{c^2}\right)$$

Or. $\qquad = x'^2 + y'^2 + z'^2 - c'^2\,t'^2 = x^2 + y^2 + z^2 - c^2t^2$

From this the required result follows.

Problem 6- Show that for low values of v, the Lorentz transformation approaches in Galilean.

Solution: Lorentz transformation equations are:

$$x' = \frac{x - vt}{\left(1 - \dfrac{v^2}{c^2}\right)^{1/2}} \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = \frac{t - (vx/c^2)}{\left(1 - \dfrac{v^2}{c^2}\right)^{1/2}}$$

If v is small, i.e., if $v << c$, so that $(v/c) \rightarrow 0$ then the foregoing equations become

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$$

Which are Galilean transformation equations.

Chapter 3

Relativistic Mechanics

1- Mass and momentum:

In classical mechanics, the linear momentum p of a moving panicle is defined as p = mv, where m is the mass and v is its velocity. In classical mechanics, we suppose that:

(i) The mass of moving body is the same as that of stationary one.

(ii) Total momentum of a body remains unchanged if no external forces are applied.

This is known as law of conservation of momentum.

The assumption (i) will not be true if we examine it by the help of Lorentz transformations.

We shall later on see that the Lorentz invariance of the law conservation of momentum implies that the mass of a moving body is not constant but it changes with velocity.

2- Newton's laws of motion:

In classical mechanics, Newton has given three laws of motion namely:

(i) A body at rest remains at rest and a body in motion continues with constant velocity in a straight line unless external force is applied to it.

Symbolically: $F = 0 \Rightarrow a = 0$

Where F and a denote respectively net external force and acceleration.

(ii) If a force F acts on a body, then the momentum of body will be changed so that rate of change of momentum proportional to the force and is in the direction of that fore Mathematically it is expressed as:

$F = K\frac{dp}{dt}$ where K is constant of proportionality. We define K . s . t. K = 1 and dimensionless.

$$\therefore F = \frac{dp}{dt}$$

In the non-relativistic limit the momentum is given by:

$$F = m\frac{dv}{dt} = ma$$

It is non-relativistic form of Newton's second law:

(iii) Whenever two bodies interact, then force $F_{1\to2}$ on the second body exerted by the first body is equal and opposite to the force $F_{2\to1}$ on the first body due lo the second. That is to say, action and reaction are equal and opposite.

$$F_{1\to2} = -F_{2\to1}$$

3- Measurement of different units:

There are three systems of units namely C.G.S. , F.P.S. , M.K.S. Note the Joule as in M.K.S. where as ergs is in C.G.S.

(1) In the formula $E = mc^2$, units of m, c and E are gram, cm./sec, ergs.

(2) 1 eV = 1 electron volt = 1.6×11^{-12} ergs.

(3) 1 Joule = 10^7 ergs.

$1 \text{ eV} = 1.6 \times 10^{-12} \times 10^{-7} = 1.6 \times 10^{-19}$ joule.

(4) 1 MeV = 10^8 eV = $1.6 \times 10^{-12} \times 10^6$ ergs.

1 BeV = 10^9 eV = $10^9 \times 1.6 \times 10^{-12}$ ergs = 1.6×10^{-3} ergs.

MeV = Million electron volt, BeV = Billion electron volt.

(5) Rest mass of proton = $m_o = 1.67 \times 10^{-24}$ gm.

(6) Rest mass of electron = $m_o = 9 \times 10^{-28}$ gram.

(7) 1 Kilowatt-hour = 1 K.W.H = 3.6×10^{13} ergs.

(8) 1 gm = 6×10^{23} a.m.u.

a.m.u. = Atomic mass unit.

(9) 1 calory = 4.2×10^7 ergs = 4.2 Joules.

(10) Distance from the earth to the sun is about 150×10^6 Km.

4- To prove that:

Variation of mass with velocity. $m = m_o \Big/ \left(1 - \frac{u^2}{c^2}\right)^{1/2}$, where u is velocity of the body when its mass is m and m_o is the mass of the body when it is at rest.

Consider two frames of reference S and S' ; S' is moving with constant velocity v along X-axis. Let m_1 be the mass of a particle moving with velocity u_1 in the system S along x-axis; the corresponding mass and velocity of the same particle in S' are m_1' and u_1' respectively.

Suppose: $\beta = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$, $\beta_1 = \frac{1}{\sqrt{\left(1 - \frac{u_1^2}{c^2}\right)}}$,

$$\beta_1' = \frac{1}{\sqrt{\left(1 - \frac{u_1'^2}{c^2}\right)}} \qquad (1)$$

By the formula of composition of velocities:

$$u_1 = \frac{u_1' + v}{1 + \frac{v}{c^2} u_1'}$$

From which: $\left(1 + \frac{v}{c^2} u_1'\right) u_1 = u_1' + v$

This gives $$u_1' = \frac{u_1 - v}{1 - \frac{v}{c^2}u_1} \tag{2}$$

$$\therefore \beta_1' u_1' = \frac{u_1 - v}{\left(1 - \frac{u_1'^2}{c^2}\right)^{1/2}\left(1 - \frac{v}{c^2}u_1\right)} \tag{3}$$

Now $$1 - \frac{u_1'^2}{c^2} = 1 - \frac{(u_1 - v)^2}{c^2\left(1 - \frac{v}{c^2}u_1\right)^2}$$

$$= \frac{c^2\left(1 - \frac{v}{c^2}u_1\right)^2 - (u_1 - v)^2}{c^2\left(1 - \frac{v}{c^2}u_1\right)^2}$$

$$\therefore c^2\left(1 - \frac{u_1'^2}{c^2}\right)\left(1 - \frac{v}{c^2}u_1\right)^2 = \left(1 - \frac{v}{c^2}u_1\right)^2 c^2 - (u_1 - v)^2$$

$$= c^2\left(1 + \frac{v^2}{c^4}u_1^2 - \frac{2u_1 v}{c^2}\right) - \left(u_1^2 + v^2 - 2u_1 v\right)$$

$$= c^2 + \frac{u_1^2 v^2}{c^2} - u_1^2 - v^2 = c^2\left(1 - \frac{u_1^2 + v^2}{c^2} + \frac{u_1^2 v^2}{c^4}\right)$$

Dividing by c^2:

$$\left(1 - \frac{u_1'^2}{c^2}\right)\left(1 - \frac{v}{c^2}u_1\right)^2 = 1 - \frac{u_1^2 + v^2}{c^2} + \frac{u_1^2 v^2}{c^4}$$

Making use of this in (3), we obtain:

$$\beta_1'u_1' = \frac{u_1 - v}{\left(1 - \frac{u_1^2 + v^2}{c^2} + \frac{u_1^2 v^2}{c^4}\right)^{1/2}} \quad (4)$$

$$= \frac{u_1 - v}{\left[\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u_1^2}{c^2}\right)\right]^{1/2}} = \beta\beta_1(u_1 - v) \quad \text{by (1):}$$

Thus $\beta_1'u_1' = \beta\beta_1(u_1 - v)$

Or $$\frac{\beta_1'u_1'}{\beta_1} = \beta(u_1 - v) \quad (5)$$

Now we suppose that there are a number of such particles moving along X-axis such that their masses and momentum are invariant in the system S so that:

$$\left.\begin{aligned} \sum m_1 &= \text{const.} \\ \sum m_1 u_1 &= \text{const.} \end{aligned}\right\} \quad (6)$$

Since β and v are the same for every particle and therefore:

$$\sum m_1 \beta v = \text{const.} \quad , \quad \sum m_1 u_1 \beta = \text{const.}$$

Subtracting, we get: $\sum m_1 \beta(u \;\; - v) = \text{const.}$

Using (5), we get:

$$\sum \left[m_1 \frac{\beta_1'u_1'}{\beta_1} \right] = \text{const.} \quad (7)$$

Applying law of conservation of momentum in S':

$$\sum m_1' u_1' = \text{const.} \qquad (8)$$

Comparing (7) and (8): $\dfrac{m_1\beta_1'}{\beta_1} = m_1'$

Or $\dfrac{m_1}{\beta_1} = \dfrac{m_1'}{\beta_1}$ = an absolute constant = m_o, say $\therefore m = \beta m_o$.

Then $$m_1 = \frac{m_o}{\sqrt{\left(1-\frac{u_1^2}{c^2}\right)}} \quad , \quad m_1' = \frac{m_o}{\sqrt{\left(1-\frac{u_1'^2}{c^2}\right)}}$$

This proves that if a particle of mass m relative to the system S, is moving with velocity u relative to the system S, then:

$$m = \frac{m_o}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}$$

If u = 0 then the last gives $m = m_o$.

Hence m_o is the mass of the body at rest. Hence mo is also called rest mass or proper mass.

For it is mass of the body measured, like proper length and proper time, in the inertial frame in which the body is at rest.

Remark: If $u << c$, then $\frac{u}{c} << 1$ so that $m \simeq m_o$

5- To show that:

Equivalence of mass of and energy $E = mc^2$.

Consider two system S and S' ; S' is moving with velocity v along X-axis relative to S. Let a particle of mass m be moving with velocity v along X-axis in the system S, then:

$$m = \frac{m_o}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \qquad (1)$$

If a force F is applied on this particle, then the increase in energy of the particle is given by $dT = F \cdot dr$ = Force × distance, dr is the displacement of the particle.

$$dT = F \cdot \frac{dr}{dt} \cdot dt = Fvdt \quad \text{or} \quad dT = Fvdt \qquad (2)$$

Force is defined as the rate of change of momentum so that:

$$F = \frac{d}{dt}(mv) \quad \text{or} \quad F\ dt = d\ (mv)$$ Now (2) becomes:

$$\therefore dT = vd = (mv)$$

$$dT = vd\left(\frac{m_o v}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}\right) = m_o v\ d\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

$$= m_o v\left(\sqrt{\left(1-\frac{v^2}{c^2}\right)}+\frac{v^2/c^2}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}\right)\frac{dv}{\left(1-\frac{v^2}{c^2}\right)}$$

$$dT = \frac{m_o v\ dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \qquad (3)$$

Taking differential of both sides in (1):

$$dm = \frac{m_o\left(-2v/c^2\right)dv}{(-2)\left(1-\frac{v^2}{c^2}\right)^{3/2}} = \frac{m_o v\ dv}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}}$$

Or

$$c^2 dm = \frac{m_o v\ dv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \qquad (4)$$

In view of this, (3) becomes $dT = c^2 dm$.

Suppose the particle is initially at rest and time its mass is m_o. After the application of the force F, its mass becomes $m_9 + dm = m$, say. The total K.E.T acquired by the particle is given by:

$$T = \int dT = \int_{m_o}^{m} c^2 dm = \int_{m_o}^{m} c^2 dm = c^2(m - m_o)$$

Or $\quad T = mc^2 - m_0c^2 \quad$ or $\quad T + m_0c^2 = mc^2$

$$T + v = mc^2 \qquad \therefore E = mc^2$$

Taking E = K.E. of the moving particle + energy at rest.

$$= T + m_0c^2 \tag{5}$$

we obtain $E = mc^2$

This formula is known as Einstein formula showing that the two fundamental conceptions of mass and energy are identical.

Here m_0c^2 is also called internal energy (or rest energy).

Deductions:

(i) We have $E = T + m_0c^2 = mc^2 + m_0c^2$, T = energy of the particle when it is in motion, m_0c^2 = Rest energy:

E = Total energy of the panicle.

$$\therefore T = E - m_0c^2 = mc^2 - m_0c^2 = m_0c^2\left[\frac{1}{(1 - v^2/c^2)^{1/2}} - 1\right]$$

Or $$T = m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right] = m_0c^2\left[\left(1 + \frac{v^2}{2c^2}.....\right) - 1\right]$$

If v/c << 1, then the last gives:

$$T = m_0c^2 \quad \frac{v^2}{2c^2} = \tfrac{1}{2}m_0v^2$$

This confirms the Newtonian limit of the last relativistic result.

(ii) $T = (m - m_o)c^2$.

This shows that a change in K. E. of a particle is relative to, a change in (inertial) mass.

(iii) It is clear from the equation:

$$T = m_o c^2 \left[\frac{1}{(1 - v^2 / c^2)^{1/2}} - 1 \right]$$

that if $v \to c$, then $T \to \infty$. It means that an infinite amount of work would need to be done on the particle to accelerate it up to the speed of light.

This we can say that:

An infinite amount of energy is needed to increase the velocity of light.

6- Transformation formula for mass:

Let us consider two systems S and S' , S' is moving with velocity v along x- along. Let m and m' be the masses of a body in S and S', which is moving with velocity u and u' in S and S' respectively.

We have $$m = \frac{m_o}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \quad , \quad m' = \frac{m_o}{\sqrt{\left(1 - \frac{u'^2}{c^2}\right)}}$$

$$u^2 = u_x^2 + u_y^2 + u_z^2 \quad , \quad u'^2 = u'^2_x + u'^2_y + u'^2_z$$

By the law of composition of velocities.

$$u_x' = \frac{u_x - v}{\left(1 - \frac{v}{c^2}u_x\right)}, \quad u_y' = \frac{u_y\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{1 - \frac{v}{c^2}u_x}, \quad u_z' = \frac{u_z\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{1 - \frac{v}{c^2}u_x}$$

From (1):

$$\frac{m}{m'} = \left(\frac{1 - \frac{u'^2}{c^2}}{1 - \frac{u^2}{c^2}}\right)^{1/2} \tag{2}$$

$$1 - \frac{u'^2}{c^2} = 1 - \frac{1}{c^2}\left(u'^2_x + u'^2_y + u'^2_z\right)$$

$$= 1 - \left[(u_x - v)^2 + u_y^2\left(1 - \frac{v^2}{c^2}\right) + u_z^2\left(1 - \frac{v^2}{c^2}\right)\right] \cdot \frac{1}{\left(1 - \frac{v}{c^2}u\right)^2 c^2}$$

$$= \frac{1}{\left(1 - \frac{v}{c^2}u_x\right)^2 c^2}\left[c^2\left(1 - \frac{v}{c^2}u_x\right)^2 - \right.$$

$$\left. - \left\{(u_x - v)^2 + u_y^2\left(1 - \frac{v^2}{c^2}\right) + u_z^2\left(1 - \frac{v^2}{c^2}\right)\right\}\right]$$

Taking $\alpha^2 = 1 \Big/ c^2\left(1 - \frac{v}{c^2}u_x\right)^2$, we get:

$$1 - \frac{u'^2}{c^2}$$

$$= \alpha^2\left[c^2\left(1+\frac{v^2}{e^4}u_x^2-\frac{2vu_x}{c^2}\right)-\left\{u^2-2vu_x-\frac{v^2}{c^2}\left(u_y^2+u_z^2\right)+v^2\right\}\right]$$

$$= \alpha^2\left[c^2\left(1-\frac{u^2}{c^2}\right)+\frac{v^2}{c^2}u^2-v^2\right]$$

$$= \alpha^2\left[c^2\left(1-\frac{u^2}{c^2}\right)+v^2\left(\frac{u^2}{c^2}-1\right)\right]=\alpha^2\left(1-\frac{u^2}{c^2}\right)\left(c^2-v^2\right)$$

Or $$1-\frac{u'^2}{c^2}=\alpha^2\ .\ c^2\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)$$

Or $$1-\frac{u'^2}{c^2}=c^3\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)\Big/c^2\left(1-\frac{v}{c^2}u_x\right)^2$$

Taking square root:

$$\left(1-\frac{u'^2}{c^2}\right)^{1/2}=\frac{\left(1-\frac{u^2}{c^2}\right)^{1/2}\left(1-\frac{v^2}{c^2}\right)^{1/2}}{\left(1-\frac{v}{c^2}u_x\right)}$$

Or $$\left(\frac{1-u'^2/c^2}{1-u^2/c^2}\right)^{1/2}=\frac{\left(1-v^2/c^2\right)^{1/2}}{\left(1-vu_x/c^2\right)}$$

Using (2), we get $$\frac{m}{m'}=\frac{\left(1-v^2/c^2\right)^{1/2}}{\left(1-vu_x/c^2\right)}$$

Or
$$m' = \frac{m\left(1 - \frac{vu_x}{c^2}\right)}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} = \frac{m\left(1 - \frac{v}{c^2}u_x\right)}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

This is the transformation formula for mass.

If $u_x = 0$, then $m' = \dfrac{m}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$

7- Transformation formula for momentum and energy:

Let us consider two system S and S' , S' is moving with velocity v along X-axis. Let m and m' be the masses of a body in S and S' ; which is moving with velocities:

$u\ (u_x, u_y, u_z)$ and $u'\ (u'_x, u'_y, u'_z)$ in S and S' respectively.

Then we have the relations:

$$m' = m\left(1 - \frac{v}{c^2}u_x\right)\beta$$

$$u_x' = \frac{u_x - v}{\left(1 - \frac{v}{c^2}u_x\right)} \quad , \quad u_y' = u_y / \beta\left(1 - \frac{v}{c^2}u_x\right)$$

$$u_z' = u_z / \beta\left(1 - \frac{v}{c^2}\right) \quad , \quad \beta = 1 \Big/ \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

The components of the momentum p are:

$p_x = mu_x$, $p_y = mu_y$, $p_z = mu_z$ in S system.

$P'_x = m'u'_x$, $p'_y = m'u'_y$, $p'_z = m'u'_z$ in S' system.

$$p'_x = m'u'_x = m\left(1 - \frac{v}{c^2}u_x\right)\beta\frac{u_x - v}{\left(1 - \frac{v}{c^2}u_x\right)} = (mu_x - mv)\,\beta$$

$$p'_x = (p_x - mv)\beta = \left(p_x - \frac{vE}{c^2}\right)\beta \qquad \text{For } E = mc^2$$

Then $$p'_x = \beta\left(p_x - \frac{vE}{c^2}\right)$$

$$p'_y = m'u'_y = m\left(1 - \frac{v}{c^2}u_x\right)\beta\frac{u_y}{\beta\left(1 - \frac{v}{c^2}u_x\right)} = mu_y = p_y$$

$$p'_z = m'u'_z = m\left(1 - \frac{v}{c^2}u_x\right)\beta\frac{u_z}{\beta\left(1 - \frac{v}{c^2}u_x\right)} = mu_z = p_z$$

$$E' = m'c^2 = m\left(1 - \frac{vu_x}{c^2}\right)\beta c^2 = \beta\,(mc^2 - mvu_x)$$

$$E' = \beta\,(E - vp_x) \quad \text{For } E = mc^2 \quad , \quad p_x = mu_x$$

Thus we have shown that:

$$p'_x = \beta\left(p_x \frac{vE}{c^2}\right) \ , \ p'_y = p_y \ , \ p'_z = p_z \ , \ \beta = 1\Big/\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$E' = \beta\,(E - vp_x)$$

These are the transformation equations for momentum. These transformation equations are exactly similar to Lorentz transformation equations if we replace:

$$x, y, z, t \text{ by } p_x, p_y, p_z, \frac{E}{c^2} \text{ respectively.}$$

Deductions:

(i) Prove that $p^2 - \frac{E^2}{c^2}$ is Lorentz invariant.

Proof. $p'^2 - \frac{E'^2}{c^2}$

$$= p'^2_x + p'^2_y + p'^2_z - \beta^2 (E - vp_x)^2 \cdot \frac{1}{c^2}$$

$$= \beta^2 \left(p_x - \frac{vE}{c^2} \right)^2 + p_y^2 + p_z^2 - \beta^2 (E - vp_x)^2 \cdot \frac{1}{c^2}$$

$$= \beta^2 \left[\left(p_x^2 + \frac{v^2 E^2}{c^4} - \frac{2vE}{c^2} p_x \right)^2 - \left(E^2 + v^2 p_x^2 - 2vEp_x \right) \frac{1}{c^2} \right] + p_y^2 + p_z^2$$

$$= p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2} \quad \text{For } \beta^2 \left(1 - \frac{v^2}{c^2} \right) = 1$$

$$= p^2 - \frac{E^2}{c^2}$$

Thus $p'^2 - \frac{E'^2}{c^2} = p^2 - \frac{E^2}{c^2}$

This proves that $p^2 - \frac{E^2}{c^2}$ is Lorentz invariant.

(ii) Prove that $p^2 - \frac{E^2}{c^2} = m_o^2 c^2$.

Proof. $p^2 - \frac{E^2}{c^2}$.

$$= p_x^2 + p_y^2 + p_z^2 - \frac{(mc^2)^2}{c^2} = (mu_x)^2 + (mu_y)^2 + (mu_z)^2 - m^2c^2$$

$$= m^2\left(u_x^2 + u_y^2 + u_z^2\right) - m^2c^2 = m^2u^2 - m^2c^2 = m^2(u^2 - c^2)$$

$$= \left[\frac{m_o}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}\right]^2 . \; c^2\left(1 - \frac{u^2}{c^2}\right) = m_o^2c^2$$

Thus $p^2 - \frac{E^2}{c^2} = m_o^2c^2$.

8- Transformation formula for force:

Let us consider two system S and S' , S' is moving with velocity v along X-axis. Let m and m' be the masses of a body in S and S' ; which is moving with velocities u and u' in S and S' respectively. If F is a force on a body of mass m and velocity u, then:

$$F = \text{rate of change of momentum} = \frac{d}{dt}(mu).$$

$$F = u\frac{dm}{dt} + m\frac{du}{dt} = u\frac{dm}{dt} + m\dot{u} \qquad (1)$$

$$F = iF_x + jF_y + kF_z \quad , \qquad u = iu_x + ju_y + ku_z$$

This gives

$$\left.\begin{aligned} F_x &= u_x\frac{dm}{dt} + m\dot{u}_x \\ F_y &= u_y\frac{dm}{dt} + m\dot{u}_y \\ F_z &= u_z\frac{dm}{dt} + m\dot{u}_z \end{aligned}\right\} \qquad (2)$$

$$\frac{dm}{dt} = \left\{\frac{m_o}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}}\right\} = \frac{u}{c^2} m_o \frac{du}{dt} \cdot \frac{I}{\left(1 - u^2/c^2\right)^{3/2}}$$

$$= u\frac{du}{dt} \cdot \frac{m}{c^2\left(1 - u^2/c^2\right)} = \frac{m}{\left(c^2 - u^2\right)} u\frac{du}{dt}$$

$$\frac{dm}{dt} = \frac{m}{\left(c^2 - u^2\right)} u\frac{du}{dt}$$

$$u^2 = u_x^2 + u_y^2 + u_z^2$$

Differentiating w . r . t. t,

$$u\ \dot{u} = u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z$$

Now (3) becomes:

$$\frac{dm}{dt} = \frac{m\left(u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z\right)}{\left(c^2 - u^2\right)}$$

Now (2) becomes:

$$\left.\begin{aligned}
F_x &= u_x \,.\, \frac{m\left(u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z\right)}{\left(c^2 - u^2\right)} + m\dot{u}_x \\
F_y &= \frac{m}{\left(c^2 - u^2\right)}\left(u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z\right)u_y + m\dot{u}_y \\
F_z &= \frac{m}{\left(c^2 - u^2\right)}\left(u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z\right)u_z + m\dot{u}_z
\end{aligned}\right\} (4)$$

By Lorentz transformation:

$$t' = \beta\left(t - \frac{vx}{c^2}\right) \text{ so that } \frac{dt'}{dt} = \beta\left(1 - \frac{v}{c^2}u_x\right)$$

Where $\beta = 1\Big/\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$

In the system S':

$$F_x' = \frac{d}{dt'}(m'u_x') \text{ according to (1).}$$

$$= \frac{d}{dt}(m'u_x') \,.\, \frac{dt}{dt'}$$

$$= \frac{d}{dt}\left[\frac{m\left(1 - \frac{v}{c^2}u_x\right)}{\sqrt{\left(1 - v^2/c^2\right)}} \,.\, \frac{(u_x - v)}{\left(1 - \frac{v}{c^2}u_x\right)}\right] \,.\, \frac{1}{\beta\left(1 - \frac{v}{c^2}u_x\right)}$$

$$= \frac{1}{\beta\sqrt{(1 - v^2/c^2)}} \frac{d}{dt}[m(u_x - v)] \cdot \frac{1}{\left(1 - \frac{v}{c^2}u_x\right)}$$

$$= \frac{1}{\left(1 - \frac{v}{c^2}u_x\right)} \left[\frac{dm}{dt}(u_x - v) + m\dot{u}_x\right]$$

$$F_x' = \frac{1}{\left(1 - \frac{v}{c^2}u_x\right)} \cdot \left[\frac{m}{(c^2 - u^2)}(u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z)(u_x - v) + m\dot{u}_x\right] \quad (5)$$

Observe that:

$$F_x - \frac{(v/c^2)(u_yF_y + u_zF_z)}{1 - \frac{v}{c^2}u_x}$$

Making use of (2), we obtain:

$$= \frac{m}{(c^2 - u^2)}(u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z)u_x + m\dot{u}_x - \frac{(v/c^2)}{1 - \frac{v}{c^2}u_x}$$

$$\left[\frac{m}{(c^2 - u^2)}(u_y^2 + u_z^2)(u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z) + m(u_y\dot{u}_y + u_z\dot{u}_z)\right]$$

$$= \frac{m}{\left(1 - \frac{v}{c^2}u_x\right)}\left[\frac{1}{(u^2 - c^2)} \cdot (u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z)\left\{\left(1 - \frac{v}{c^2}u_x\right)u_x\right.\right.$$

$$-\frac{v}{c^2}\left(u_y^2+u_z^2\right)\Big\}-\left(u_y\dot{u}_y+u_z\dot{u}_z\right)\frac{v}{c^2}+\left(1-\frac{v}{c^2}u_x\right)\dot{u}_x\Big]$$

$$=\frac{m}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{\left(u_x\dot{u}_x+u_y\dot{u}_y+u_z\dot{u}_z\right)\left(u_x-\frac{v}{c^2}u^2\right)}{\left(c^2-u^2\right)}\right.$$

$$\left.-\frac{v}{c^2}\left(u_x\dot{u}_x+u_y\dot{u}_y+u_z\dot{u}_z\right)+\dot{u}_x\right]$$

$$=\frac{1}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{m}{\left(c^2-u^2\right)}\left(u_x\dot{u}_x+u_y\dot{u}_y+u_z\dot{u}_z\right)\left\{\left(u_x-\frac{v}{c^2}u^2\right)\right.\right.$$

$$\left.\left.-\left(c^2-u^2\right)\frac{v}{c^2}\right\}+m\dot{u}_x\right]$$

$$=\frac{1}{\left(1-\frac{v}{c^2}u_x\right)}\left[\frac{m}{\left(c^2-u^2\right)}\cdot\left(u_x\dot{u}_x+u_y\dot{u}_y+u_z\dot{u}_z\right)\left\{u_x-v\right\}+m\dot{u}_x\right] \tag{6}$$

$= F_x'$, by (5):

Thus $$F_x' = F_x - \frac{(v/c^2)(u_yF_y+u_zF_z)}{\left(1-\frac{v}{c^2}u_x\right)}$$

By virtue of (1), we have:

$$F_y' = \frac{d}{dt'}\left(m'u_y'\right) = \frac{d}{dt}\left(m'u_y'\right)\cdot\frac{dt}{dt'}$$

$$= \frac{1}{\beta\left(1-\frac{v}{c^2}u_x\right)} \frac{d}{dt}\left(\frac{m\left(1-\frac{v}{c^2}u_x\right)}{\sqrt{(1-v^2/c^2)}} \cdot u_y \frac{\sqrt{(1-v^2/c^2)}}{1-\frac{v}{c^2}u_x} \right)$$

$$= \frac{\sqrt{(1-v^2/c^2)}}{1-\frac{v}{c^2}u_x} \frac{d}{dt}(m\ u_y) = \frac{\sqrt{(1-v^2/c^2)}}{1-\frac{vu_x}{c^2}} \cdot F_y$$

Thus we have shown that:

$$F_x' = F_x - \frac{(v/c^2)}{1-\frac{v}{c^2}u_x}(u_y F_y + u_z F_z)$$

$$F_y' = \frac{\sqrt{(1-v^2/c^2)}}{1-\frac{v}{c^2}u_x} F_y \quad , \quad F_z' = \frac{\sqrt{(1-v^2/c^2)}}{1-\frac{v}{c^2}} F_z$$

These are the required transformation formulae for a force acting on a body.

Remark: The inverse formulae are:

$$F_x = F_x' + \frac{(v/c^2)}{1+\frac{v}{c^2}u_x'}(u_y' F_y' + u_z' F_z')$$

$$F_y = F_y' \frac{\sqrt{(1-v^2/c^2)}}{1-\frac{v}{c^2}u_x'} \quad , \quad F_z = F_z' \frac{\sqrt{(1-v^2/c^2)}}{1-\frac{v}{c^2}u_x'}$$

9- Relativistic Equations:

We know that the relativistic kinetic energy of a particle of rest mass m_o moving with velocity $u = c\,\beta$ is:

$$T = mc^2 - m_o c^2$$

$$T = m_o c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) \qquad (1)$$

We note that $\dfrac{\partial T}{\partial x}$ does not give the correct relativistic momentum:

$$P_x = \frac{m_o \dot{x}}{\sqrt{1-\beta^2}}$$

We therefore define a function T^* such that:

$$P_x = \frac{m_o \dot{x}}{\sqrt{1-\beta^2}} = \frac{\partial T^*}{\partial \dot{x}} \qquad (2)$$

Similarly for y and z components.

Thus we have:

$$dT^* = \frac{\partial T^*}{\partial x} dx^{\bullet} + \frac{\partial T^*}{\partial y} dy^{\bullet} + \frac{\partial T^*}{\partial z} dz^{\bullet}$$

$$= \frac{m_o}{\sqrt{1-\beta^2}}\ (x^{\bullet}dx^{\bullet} + y^{\bullet}dy^{\bullet} + z^{\bullet}dz^{\bullet})$$

$$= \frac{m_o}{\sqrt{1-\beta^2}} u\,du. \quad \text{For} \quad u^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$= \frac{m_o c^2 \beta\, d\beta}{\sqrt{1-\beta^2}} \qquad (3)$$

Since $\quad u = c\,\beta$

Integrating we get:

$$T^* = m_o c^2 \sqrt{1-\beta^2} + A$$

To find A, we use the condition that when u is very small compared to c , T^* must be equal to $\frac{1}{2} m_o c^2$.

Hence $\quad \frac{1}{2} m_o u^2 = -m_o c^2 (1 - \frac{1}{2}\beta^2) + A$

$$= -m_o c^2 + \frac{1}{2} m_o u^2 + A$$

$$\therefore A = +m_o c^2$$

Hence $\quad T^* = m_o c^2 (1 - \sqrt{1-\beta^2}) \qquad (4)$

The relativistic Lagrangian L can now be defined as:

$$L = T^* - V = m_o c^2 (1 - \sqrt{1-\beta^2}) - V \qquad (5)$$

Lagrange's equation are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \text{etc.}$$

But $$\frac{\partial L}{\partial x^{\bullet}}=\frac{\partial T^{\bullet}}{\partial x}=\frac{m_o x}{\sqrt{1-\beta^2}}$$

Since $$\frac{\partial V}{\partial x}=0$$

$$\frac{\partial L}{\partial x}=-\frac{\partial V}{\partial x}$$

Hence the equation are:

$$\frac{d}{dt}\frac{m_o x^{\bullet}}{\sqrt{1-\beta^2}}=-\frac{\partial V}{\partial x}$$

$$\frac{d}{dt}\frac{m_o y^{\bullet}}{\sqrt{1-\beta^2}}=-\frac{\partial V}{\partial y} \qquad (6)$$

$$\frac{d}{dt}\frac{m_o z^{\bullet}}{\sqrt{1-\beta^2}}=-\frac{\partial V}{\partial z}$$

Which are the relativistic equations of motion.

Thus $$L=m_o c^2(1-\sqrt{1-\beta^2})-V$$

Gives the correct relativistic equations of motion.

We now define the relativistic Hamiltonian as:

$$H=\Sigma p_x\, x^{\bullet}-L$$

$$=\Sigma\frac{\partial T}{\partial x^{\bullet}}\,.\,x^{\bullet}-m_o c^2(1-\sqrt{1-\beta^2})+V \qquad (7)$$

$$= \frac{m_o x^\bullet}{\sqrt{1-\beta^2}} \; x^\bullet - m_o c^2(1-\sqrt{1-\beta^2}) + V$$

$$= \frac{m_o}{\sqrt{1-\beta^2}}(x^{\bullet 2} + y^{\bullet 2} + z^{\bullet 2})$$

$$- m_o c^2(1-\sqrt{1-\beta^2}) + V$$

$$= \frac{m_o u^2}{\sqrt{1-\beta^2}} \; - m_o c^2(1-\sqrt{1-\beta^2}) + V$$

$$= \frac{m_o c^2 \beta^2}{\sqrt{1-\beta^2}} \; - m_o c^2(1-\sqrt{1-\beta^2}) + V$$

$$= m_o c^2(1-\sqrt{1-\beta^2}) + V$$

$$= T + V \qquad (8)$$

So that H is the total energy.

Also $\qquad P_x = \frac{m_o \dot{x}}{\sqrt{1-\beta^2}}$ etc.

$$\therefore P_x^2 + P_y^2 + P_z^2 = \frac{m_o^2}{1-\beta^2}(x^{\bullet 2} + y^{\bullet 2} + z^{\bullet 2}) = \frac{m_o^2 c^2 \beta^2}{1-\beta^2}$$

$$\therefore \frac{1}{m_o^2 c^2}(P_x^2 + P_y^2 + P_z^2) = \frac{\beta^2}{1-\beta^2}$$

$$= \frac{\beta^2}{1-\beta^2} = \frac{1}{1-\beta^2} - 1$$

$$\therefore \frac{1}{1-\beta^2} = \frac{m_o^2c^2 + P_x^2 + P_y^2 + P_z^2}{m_o^2c^2}$$

$$H = c\sqrt{m_o^2c^2 + P_x^2 + P_y^2 + P_z^2} - m_oc^2 + V \qquad (9)$$

Now Hamilton's canonical equations are:

$$x^{\bullet} = \frac{\partial H}{\partial P_x}$$

And $$P_x^{\bullet} = \frac{\partial H}{\partial x} \qquad (10)$$

Giving $$x^{\bullet} = \frac{cP_x}{\sqrt{m_o^2c^2 + P_x^2 + P_y^2 + P_z^2}}$$

$$= \frac{cP_x\sqrt{1-\beta^2}}{m_oc}$$

$$P_x = \frac{m_ox^{\bullet}}{\sqrt{1-\beta^2}}$$

$$\frac{d}{dt}\left(\frac{m_ox^{\bullet}}{\sqrt{1-\beta^2}}\right) = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} \qquad (11)$$

etc.

Which are the correct relativistic equations. Thus (9) gives the correct Hamiltonian suitable for relativistic equations.

8- Kepler's motion:

The kinetic energy is: .

$$T = m_o c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) \tag{1}$$

and the Lagrangian function is:

$$L = m_o c^2(1 - \sqrt{1-\beta^2}) - V \tag{2}$$

where V is the potential energy:

$$V = -\frac{\mu}{r} \quad \text{from the inverse square law.}$$

and $$\beta^2 = \frac{u^2}{c^2} = \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2} \text{ in polar coordinates.}$$

The Lagrange's θ equation is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

Now $$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \beta}\frac{\partial \beta}{\partial \dot{\theta}} = \frac{m_o c^2 \beta}{\sqrt{1-\beta^2}}\frac{\partial \beta}{\partial \dot{\theta}}$$

$$= \frac{m_o c^2 \beta}{\sqrt{1-\beta^2}}\frac{r^2\dot{\theta}}{c^2\beta}$$

$$\therefore \frac{d}{dt}\left(\frac{m_o}{\sqrt{1-\beta^2}} r^2\dot{\theta}\right) = 0$$

$$\therefore \frac{m_o r^2\dot{\theta}}{\sqrt{1-\beta^2}} = \text{const.} = p \text{ say} \tag{3}$$

Equation of energy is T + V = E where E is the total energy:

$$\therefore m_o c^2\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)-\frac{\mu}{r}=E$$

Or
$$\frac{1}{1-\beta^2}=\left(1+\frac{E+\frac{\mu}{r}}{m_o c^2}\right)^2 \tag{4}$$

Now p_r = momentum in r-direction:

$$=\frac{m_o r^{\bullet}}{\sqrt{1-\beta^2}} \tag{5}$$

Also
$$\beta^2=\frac{r^{\bullet 2}+r^2\dot{\theta}^2}{c^2}$$

$$=\frac{(1-\beta^2)P_r^2+\frac{p^2}{r^2}(1-\beta^2)}{m_o^2c^2} \qquad \text{from (3), (5)}$$

Or
$$\frac{\beta^2}{1-\beta^2}=\frac{P_r^2+\frac{p^2}{r^2}}{m_o^2c^2}$$

Or
$$\frac{1}{1-\beta^2}=1+\frac{1}{m_o^2c^2}\left(P_r^2+\frac{p^2}{r^2}\right)$$

$$= 1 + \frac{p^2}{m_o^2 c^2}\left(\frac{P_r^2}{p^2} + \frac{1}{r^2}\right)$$

Hence
$$1 + \frac{p^2}{m_o^2 c^2}\left(\frac{P_r^2}{p^2} + \frac{1}{r^2}\right)$$

$$= \left(1 + \frac{E + \frac{\mu}{r}}{m_o c^2}\right)^2 \qquad (6)$$

Now
$$\frac{p_r}{p} = \frac{\dot{r}}{r^2\dot{\theta}} = \frac{dr}{r^2 d\theta} = \frac{du}{d\theta} \quad \text{where } u = \frac{1}{r}$$

$$1 + \frac{p^2}{m_o^2 c^2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$= \left(1 + \frac{E + \frac{\mu}{r}}{m_o c^2}\right)^2 . \qquad (7)$$

Differentiating w . r . t. θ we get:

$$\frac{p^2}{m_o^2 c^2}\left[2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta}\right]$$

$$= 2\left(1 + \frac{E + \mu u}{m_o c^2}\right) \frac{\mu}{m_o c^2}\frac{du}{d\theta}$$

Or $$\frac{d^2u}{d\theta^2}+u=\frac{m_o\mu}{p^2}\left(1+\frac{E}{m_oc^2}+\frac{\mu}{m_oc^2}u\right) \quad (8)$$

Or $$\frac{d^2u}{d\theta^2}+u\left(1-\frac{\mu^2}{p^2c^2}\right)-\frac{m_o\mu}{p^2}\left(1+\frac{E}{m_oc^2}\right)=0 \quad (9)$$

Or $$\frac{d^2u}{d\theta^2}+k^2(u-\varepsilon)=0 \quad (10)$$

Where $$k^2=1-\frac{\mu^2}{p^2c^2} \quad , \quad k^2\varepsilon=\frac{m_o\mu}{p^2}\left(1+\frac{E}{m_oc^2}\right) \quad (11)$$

The solution of this equation is:

$$u-\varepsilon=A\cos k\theta+B\sin k\theta$$

Or $$u=\varepsilon+A\cos k\theta+B\sin k\theta \quad (12)$$

If A be the initial position of the perihelion i.e.

When $$\theta=0=\frac{du}{d\theta}$$

Then $$B=0$$

$$\therefore u=\varepsilon+A\cos k\theta$$

Or $$\frac{1}{r}=\varepsilon+A\cos k\theta \quad (13)$$

This equation differs from the classical equation in having $k\theta$ in place of θ.

Now $$k^2=1-\frac{p_o^2}{p^2} \quad \text{where } p_o=\frac{\mu}{c} \quad (14)$$

When $c \to B$, $p_o = 0$

$$\therefore k = 1$$

So that the equation (13) is the ordinary kepler ellipse.

The orbit raches is perihelion next when:

$$\frac{du}{d\theta} = 0$$

i.e. when $k\theta = 2\pi$

i.e. $\theta = \frac{2\pi}{k} > 2\pi$

The motion of the perihelion occurs in the same sense in which the orbit is being described and has the angular velocity:

$$\Delta\theta = \frac{2\pi}{k} - 2\pi \tag{15}$$

The planet Mercury exhibits the advance of the perihelion amounting to 43" per century. In the above formula, taking:

$$\mu = \gamma M m_o$$

where M is the mass of the sun and m_o the rest mass of Mercury and γ the constant of gravitation, we have:

$$p_o = \frac{\gamma M m_o}{c} \tag{16}$$

and hence can be calculated.

Also p, which is the moment of momentum about the center of force, can be found. Hence k can be found out. It is seen that the above formula (15) gives a value, about 7" per century which is too small.

It was only when Einstein proposed the General Theory of Relativity, which included gravitation; he was able to account for the observed motion of the perihelion of Mercury.



Chapter 4

Minkowski space

According to Minkowski, the external world is not Euclidean space of three dimensions, i.e., it is not composed of points whose co-ordinates are (x , y , z), where x , y , z, are real numbers; but it is composed of events whose co-ordinates are (x_1 , x_2 , x_3 , x_4) where the first three (x_1 , x_2 , x_3) are space co-ordinates and the fourth one x_4 in volvs time. If an event occurs in the space, then the position of the point where it occurs and the instant when it occur both are represented by the location of the event in the four dimensional continuum. All the four directions are not equivalent. For an axis, which measures the distance in x-direction, can be rotated to measure the distances in y and z directions; but the same axis can not be rotated to measure the time interval due to the fact that a meter stick can not be converted into a clock. Therefore the direction of time interval is not unique. This distinction is expressed by saying that the space time continuum is 3 + 1 dimensional rather than as four dimensional.

The internal between two events whose co-ordinates are:

$$(x , y , z , x_4) \text{ and } (x + dx , y + dy , z + dz , x_4 + dx_4)$$

is given by:

$$ds^2 = dx^2 + dy^2 + dz^2 + dx_4^2 \qquad (1)$$

where the co-ordinate x_4 involves t. This interval must be independent of transformation from one system to another system.

We have seen that the internal:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

is Lorentz invariant Consequently the invariant interval between two neighboring points must be of the form:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

Comparing (1) and (2), we get $x_4 = ict$, where $i = \sqrt{(-1)}$.

Hence the four co-ordinates of an event in the space are:

$$x_1 = x \ , \ x_2 = y \ , \ x_3 = z \ , \ x_4 = ict$$

In Newtonian Physics, an event may be identified by four members (x , y , z , t) where (x , y , z) are the rectangular. Cartesian co-ordinates of the place where it occurs and t the time at which it occurs. Hence it is clear that an event needs four numbers to identify it and for this reason we say that in Newtonian Physics to totality of all possible events form a four dimensional continuum. This continuum is called space-time; and we are not in a position to remove the hyphen and speak of

space and time respectively. Thus the co-ordinates of an event may also be taken as (x_1, x_2, x_3, x_4), where $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = t$.

1- Space and time like intervals:

Consider two frames of reference S and S' ; S' is moving with constant velocity v along X- axis. By Lorentz transformations:

$$x' = \beta\,(x - vt) \quad , \quad y' = y \quad , \quad z' = z \ ,$$

$$t' = \beta\left(t - \frac{vx}{c^2}\right) \quad , \quad \beta = 1\Big/\sqrt{\left(1 - \frac{v^2}{c^2}\right)} \qquad (1)$$

Consider two events whose co-ordinates are (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) in S.

Write:

$$s_{12}^2 = -[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] + c^2(t_2 - t_1)^2 \qquad (2)$$

In S' system, this is transformed to:

$$s'^2_{12} = -[(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2] + c^2(t'_2 - t'_1)^2 \qquad (3)$$

Writing (3) with the help of (1), we have:

$$s'^2_{12} = -\ [\beta^2\{(x_2 - x_1) - v(t_2 - t_1)\}^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]$$

$$+ c^2\beta^2\left\{(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right\}^2$$

$$= - [\beta^2(x_2 - x_1)^2\left(1 - \frac{v^2}{c^2}\right) + (y_2 - y_1)^2 + (z_2 - z')^2]$$

$$+ c^2\beta^2(t_2 - t_1)^2\left(1 - \frac{v^2}{c^2}\right) - 2v\beta^2(x_2 - x_1)\ (t_2 - t_1)$$

$$+ \frac{2v}{c^2}\beta^2c^2(x_2 - x_1)\ (t_2 - t_1)$$

But $\quad \beta^2\left(1 - \frac{v^2}{c^2}\right) = 1$

Hence the last gives:

$s'^2_{12} = -[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] + c^2(t_2 - t_1)^2$

$= s^2_{12}$, according to (2):

Thus $s'^2_{12} = s^2_{12}$ or $s'_{12} = s_{12}$

This proves that the interval s_{12} is Lorentz invariant.

Consequent we have the following result. The space-time, interval between two events is an invariant.

(i) It $s_{12} = 0$, then the interval s_{12}, given by (2), is called Singular. Also $s_{12} = 0$ gives:

$-[(x_2 - x_1)^2 + (y_2 - y_1)^2 - (z_2 - z_1)^2] + c^2(t_2 - t_1)^2 = 0$

This suggests that:

$$-[dx^2 + dy^2 + dz^2] + c^2dt^2 = 0$$

This is known as the equation of null cone or light cone.

(ii) Let the two events occur at the same point in S' and also let the first event occur after the second event so that:

$$x'_2 = x'_1 \ , \ y'_2 = y'_1 \ , \ z'_2 = z'_1 \ , \ t'_2 > t'_1$$

Putting these values in (3):

$$s'^2_{12} = c^2(t'_2 - t'_1)^2 > 0$$

$$s'^2_{12} > 0 \qquad \text{or} \qquad s_{12} > 0$$

But $s_{12} = s'_{12}$. Hence $s_{12} > 0$.

$s_{12} > 0 \Rightarrow$ the interval s_{12} is real.

Real intervals are called times like intervals:

For s'_{12} contains only time component. The condition that the interval is time like interval is:

$$c^2(t_2 - t_1)^2 > (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

If the interval is time like, then there exists a frame of reference for which the interval between two events is real.

(iii) Next we suppose that the two events in S' occur at the same time so that $t'_1 = t'_2$. Now (3) becomes:

$$s'^2_{12} = -[(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2] < 0$$

or $\qquad s'^2_{12} > 0 \qquad$ or $\qquad s^2_{12} > 0$

or s_{12} is imaginary.

Imaginary intervals are called Space like intervals. For s'_{12} contains only space co-ordinates. The condition that an interval Is space like interval is:

$$c^2(t_2 - t_1)^2 < (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Thus if the interval is space like, then there exists a frame of reference in which two events occur at the same time.

A vector is said to be time-like if its magnitude is real. It is space-like if its magnitude is imaginary. It is null if its magnitude is zero.

2- World Points and World lines:

The events in the four dimensional space or Minkowski space are represented by points known as world points and each particle corresponds to a certain line known as world line.

Without loss of generality, we consider only one space axis, the y-axis. Then the co-ordinates (x , t) of an event can be represented on a space-time diagram in which space axis is horizontal and time axis is vertical s . t. These two axes are orthogonal to each other. The dimensions of co-ordinates can be kept the same if we take ct (= m , say) instead of t. The Lorentz transformation equations for x and t are:

$$x' = \frac{x - \beta m}{\sqrt{(1 - \beta^2)}} \qquad (1)$$

$$m' = \frac{x - \beta x}{\sqrt{(1-\beta^2)}} \qquad (2)$$

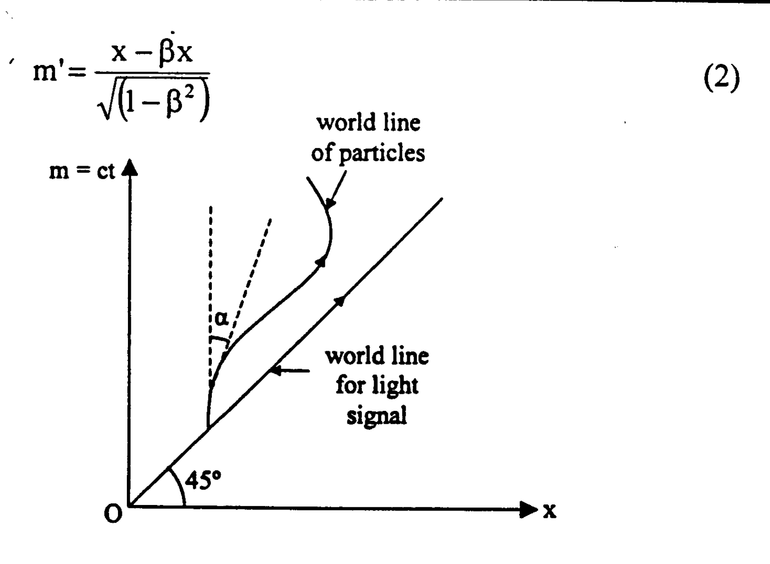


Fig. 1.

$$x = \frac{x' + \beta m'}{\sqrt{(1-\beta^2)}}, \qquad m = \frac{m' + \beta' x}{\sqrt{(1-\beta^2)}} \text{ where } \beta = v/c$$

We take x-axis to be horizontal and m to be vertical in the frame S. The path of a particle in this frame will represents a curve known as world line as shown in the diagram on page 74. The inclination a of a tangent at any point of the world line is given by:

$$\tan\alpha = \frac{1}{c}\frac{dx}{dt} = \frac{v}{c}$$

where α is an angle between tangent and m-axis. Also $\alpha < 45°$ as $v < c$ for any material particle. The world line for light signal {i.e., $v = c$) is a straight line making an angle 45° with m-axis.

A collision between two particles corresponds to an intersection of their world lines. It is quite obvious that the world line of a material particle is determined by an event and by a space-time diagram at that event. A collision is said to be elastic if the final velocity has the same magnitude as the initial velocity but opposite in directions.

3- Light cone:

The quantity $s^2 = x^2 + y^2 + z^2 - c^2t^2$ (1)

remains invariant under Lorentz transformations. Here we take:

$$x^1 = x \ , \ x^2 = y \ , \ x^3 = z \ , \ x^4 = ict = ct\sqrt{(-1)}$$

The invariant form (1) is naturally called the square of the four dimensional distance between the event (x^i) and the origin (0 , 0 , 0 , 0). All the points whose distance from the origin is zero form a surface described by the equation

$$s^2 = x^2 + y^2 + z^2 - c^2t^2 = 0 \qquad (2)$$

This surface is called light cone. For the equation (2) describes the propagation of a spherical light wave starting from the origin (0 , 0 , 0) at $t = 0$. The light cone divides the (3 + 1) space into two invariant separate domains S_1 and S_2 defined by the inequalities:

$$s^2 = x^2 + y^2 + z^2 - c^2t^2 < 0 \qquad (3)$$

and $$s^2 = x^2 + y^2 + z^2 - c^2t^2 > 0 \tag{4}$$

respectively. In the domain S_2, we have the case of Simultaneity relative to Lorentz transformations. Such a transformation is not possible for two events in the domain S_1.

4- Proper time:

Sometimes it is convenient to work with the invariant $dT^2 = ds^2/c^2$ rather than with ds^2 itself. So we give a special symbol dT and call it the element of proper time.

$$dT^2 = \frac{1}{c^2} ds^2 = -\frac{1}{c^2}(dx^2 + dy^2 + dz^2) + dt^2$$

Or $$dT^2 = dt\left[1 - \frac{1}{c^2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}\right]$$

If the particle has a velocity u, then the last becomes:

$$dT = dt\left[1 - \frac{u^2}{c^2}\right]^{1/2}$$

Integral of proper time along a world line is:

$$T = \int \left(1 - \frac{u^2}{c^2}\right)^{1/2} dt$$

5- Energy moment four vector:

To explain the formulation of energy-momentum vector in special relativity.

In classical mechanics, momentum p is defined as p = mv a similar way, the relativistic definition of momentum p is:

$$p = m_o v \Big/ \left(1 - \frac{u^2}{c^2}\right)^{1/2} \qquad (1)$$

We known that:

$$ds^2 = -(dx^2 + dy^2 + dz^2) + c^2 dt^2$$

This $\Rightarrow \left(\frac{ds}{dt}\right)^2 = -\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] + c^2 = c^2 - v^2$

$$\Rightarrow \frac{ds}{c} = dt\sqrt{\left(1 - \frac{v^2}{c^2}\right)} \text{ Write } \beta = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Then $\frac{ds}{c} = \frac{dt}{\beta}$ or $\beta ds = cdt$ (2)

Writing Cartesian equivalent of (1), We have:

$$p_x = \frac{m_o v_x}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \beta m_o v_x = \beta m_o \frac{dx}{dt} = m_o \frac{cdx}{ds}$$

By (2) or:

$$p_x = m_o c \frac{dx}{ds}$$

Similarly: $p_y = m_o c \frac{dy}{ds}$, $p_z = m_o c \frac{dz}{ds}$

$$E = mc^2 = \frac{m_o c^2}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \beta m_o c^2 = m_o c^2 \ . \ \frac{cdt}{ds}$$

Or $\quad (E/c) = m_o c \ . \ \dfrac{cdt}{ds}$

Write $\quad x^1 = x \ , \ x^2 = y \ , \ x^3 = z \ , \ x^4 = ct \quad$ we get:

$$p_x = m_o c \frac{dx^1}{ds} \ , \ p_y = m_o c \frac{dx^2}{ds} \ , \ p_z = m_o c \frac{dx^3}{ds} \ , \ \frac{E}{c} = m_o c \ . \ \frac{dx^4}{ds}$$

From this it is clear that the set of four quantities (p_x, p_y, p_z, $\frac{E}{c}$) are the components of the energy momentum four-vector and we denote this by p^μ. i.e.,

$$p^\mu = (p_x, p_y, p_z, \frac{E}{c})$$

This is the significance of the fourth component of momentum.

Remark. We now define the four-velocity of a particle as:

$$u^\mu = \frac{dx^\mu}{ds}$$

In similar way we also define four-acceleration as:

$$f^\mu = \frac{du^\mu}{ds} = \frac{d^2 x^\mu}{ds^2}$$

6- Relativistic equations of Motion:

Newton's equations for momentum and motion are taken over in the relativity theory with this difference that a body of mass m moving with velocity v satisfies the equation $m = \frac{m_o}{\sqrt{(1-\beta^2)}}$

where $\beta = v/c$. Momentum p is defined as $p = mv = \frac{m_o v}{\sqrt{(1-\beta^2)}}$.

Hence the components of momentum are:

$$p_x = \frac{m_o \dot{x}}{\sqrt{(1-\beta^2)}} \quad , \quad p_y = \frac{m_o \dot{y}}{\sqrt{(1-\beta^2)}} \quad , \quad p_z = \frac{m_o \dot{z}}{\sqrt{(1-\beta^2)}}$$

Where $\beta^2 = \frac{v^2}{c^2} = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}$

Here dot denotes differentiation w . r . t. time t. The equations of motion are:

$$\frac{dp_x}{dt} = F_x \quad , \quad \frac{dp_y}{dt} = F_y \quad , \quad \frac{dp_z}{dt} = F_z$$

i.e. $$m_o \frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{(1-\beta^2)}}\right] = F_x \; , \; m_o \frac{d}{dt}\left[\frac{\dot{y}}{\sqrt{(1-\beta^2)}}\right] = F_y ,$$

$$m_o \frac{d}{dt}\left[\frac{\dot{z}}{\sqrt{(1-\beta^2)}}\right] = F_z$$

7- Minkowski's equation of motion:

The equation:

$$\dot{p}^\mu = K^\mu \quad , \quad \text{i.e.} \quad \frac{d}{ds}\left(m_o c \frac{dx^\mu}{ds}\right) = K^\mu \qquad (1)$$

is referred to as Minkowski's equation of motion and the force K^μ is four-force. This K^μ is also known as Minkowski's force.

To show that Minkowski's equation reduces to the Newtonian form in the limit where $v/c \to 0$.

(1) is expressible as:

$$\frac{d}{dt}\left(m_o c \frac{dx^\mu}{dt}\frac{dt}{ds}\right)\frac{dt}{ds} = K^\mu$$

Or $$\frac{d}{dt}\left(m_o c \frac{dt}{ds}\frac{dx^\mu}{dt}\right) = K^\mu \frac{ds}{dt} \qquad (2)$$

But $$ds^2 = -[(dx)^2 + (dy)^2 + (dz)^2] + c^2 dt^2$$

Or $$\left(\frac{ds}{cdt}\right)^2 = 1 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \cdot \frac{1}{c^2} = 1 - \frac{v^2}{c^2}$$

Or $$\frac{1}{c} \cdot \frac{ds}{dt} = \sqrt{\left(1 - \frac{v^2}{c^2}\right)} \qquad (3)$$

$$\therefore m_o c \frac{dt}{ds} = m_o c \cdot \frac{1}{c\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = \frac{m_o}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} = m \qquad (4)$$

(according the law of variation of mass with velocity).

Writing (2) with the help of (3) and (4):

$$\frac{d}{dt}\left(m\frac{dx^{\mu}}{dt}\right)=K^{\mu}\ .\ c\sqrt{\left(1-\frac{v^2}{c^2}\right)}$$

Taking limit as $v/c \to 0$, we get:

$$\frac{d}{dt}\left(m\frac{dx^{\mu}}{dt}\right)=cK^{\mu} \quad \text{For } \mu=1\,,2\,,3\,,4.$$

Hence, in particular:

$$\frac{d}{dt}\left(m\frac{dx^{i}}{dt}\right)=cK^{i} \quad \text{For } i=1\,,2\,,3\,,4.$$

This is Newton's form of equation of motion.

Problem: Derive the fore-vector equation of motion and discuss the physical significance of the force four-vector in terms of the classical quantities.

Applications:

1- The 4-dimensionel volume element:

In the 4-dimensionel spaces we can write the 4-dimensionel volume element as fallows:

$$dV^{*}=dx_1\ dx_2\ dx_3\ dx_4=dX \quad , \quad =1\,,2\,,3\,,4$$

Or $\quad dV^{*}=dV\ dx_4 \quad$ where $dV=dx_1\ dx_2\ dx_3=dx\ dy\ dz$

But $\quad x_4 = ict \qquad \therefore dx_4 = ic\ dt$

$$\therefore dV^* = ic\ dV\ dt$$

Which is invariant with respect to Lorentz transformation.

Since $\quad dV = dx\ dy\ dz$

$$dx = d\ell = d\ell'\ \sqrt{1-\beta^2} = dx'\sqrt{1-\beta^2}$$

and $\quad \dfrac{dt'}{\sqrt{1-\beta^2}}$

$$\therefore dV^* = ic\ dx'\sqrt{1-\beta^2}\ .\ dy'dz'\ .\ \frac{dt'}{\sqrt{1-\beta^2}}$$

$$= ic\ dx'dy'dz'dt' = ic\ dV'dt' = dV^{*\prime}$$

where $\quad dV = dx\ dy\ dz = dx'\sqrt{1-\beta^2}\ dy'dz' = dV'\sqrt{1-\beta^2}$

or $\quad dV = dV_o\sqrt{1-\beta^2}$, dV_o is the proper volume, and

$dt = \dfrac{dt'}{\sqrt{1-\beta^2}}$ or $dt = \dfrac{dT}{\sqrt{1-\beta^2}}$, dT is the proper time.

$dV^* = ic\ dV\ dt = ic\ dV_o dT = inv.$

2- The 4-dimensionel operator nable(∇_k):

Let we have a scalar function of x_k:

i.e. $\quad \phi\ (x_k) \quad$ or $\quad \phi\ (x_1, x_2, x_3, x_4)$

$$d\phi \; \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 + \frac{\partial \phi}{\partial x_4} dx_4$$

which is invariant.

The quantities $\frac{\partial \phi}{\partial x_1}$, $\frac{\partial \phi}{\partial x_2}$, $\frac{\partial \phi}{\partial x_3}$, $\frac{\partial \phi}{\partial x_4}$ under a transformation of coordinates seems to be covariant components of a vector, which is called 4-gradiant of the function ϕ. Introducing the 4-operator nabla (or ∇) which have the covariant components:

$$\nabla_1 = \frac{\partial}{\partial x_1} \; , \; \nabla_2 = \frac{\partial}{\partial x_2} \; , \; \nabla_3 = \frac{\partial}{\partial x_3} \; , \; \nabla_4 = \frac{\partial}{\partial x_4}$$

or

$$\nabla_k = \left(\bar{\nabla} \; , \; \nabla_4\right) = \left(\bar{\nabla} \; , \; \frac{\partial}{\partial x_4}\right) = \frac{\partial}{\partial x_k}$$

where $\bar{\nabla}$ is the operator grad in the 3-dimensional Euclidean space.

But $\nabla_4 = ict$, $\therefore \frac{\partial}{\partial x_4} = \frac{1}{ic}\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial t} = ic\frac{\partial}{\partial x_4}$

$$\nabla_k = \frac{\partial}{\partial x_k} = \left(\bar{\nabla} \; , \; \frac{1}{ic}\frac{\partial}{\partial t}\right) = \left(\bar{\nabla} \; , \; -\frac{i}{c}\frac{\partial}{\partial t}\right)$$

The product of two operators ∇_k and ∇_k is then:

$$\nabla_k^2 = (\nabla_k, \nabla_k) = \vec{\nabla} \cdot \vec{\nabla} + \nabla_4 \nabla_4 = \nabla^2 + \left(\frac{-i}{c}\frac{\partial}{\partial t}\right)\left(\frac{-i}{c}\frac{\partial}{\partial t}\right)$$

$\nabla_k^2 = \nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}$ which is the 4-dinenaional.

form of the Laplacian operator and known as the D'Alembertian operator.

3- Transformation equations in 4-dimensional space:

In the 4-dimensional space the covariant component a of the co-ordinate 4-vector are:

$$x_k = (x_1, x_2, x_3, x_4)$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$.

and in transforming from one co-ordinate system S' to another system S these components art transformed as follows:

$$x'_k = \sum_{\gamma=1}^{4} a_{k\gamma}\, x_\gamma = a_{k\gamma}\, x_\gamma \qquad \text{(I)}$$

where $a_{k\gamma}$ is the coefficients of the transformation, and we can write it as a matrix.

$$a_{k\gamma} = \begin{bmatrix} \alpha & 0 & 0 & i\beta\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\alpha & 0 & 0 & \alpha \end{bmatrix} \qquad \text{(II)}$$

where $\alpha = \dfrac{1}{\sqrt{1-\beta^2}}$, $\beta = \dfrac{V}{c}$

the inverse transformation is:

$$x_k = \sum_{\gamma=1}^{4} a'_{k\gamma}\ x'_\gamma = a'_{k\gamma}\ x'_\gamma \qquad \text{(III)}$$

where $a'_{k\gamma}$ is the matrix of transformation and has the form:

$$a'_{k\gamma} = \begin{bmatrix} \alpha & 0 & 0 & i\beta\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\alpha & 0 & 0 & \alpha \end{bmatrix} \qquad \text{(IV)}$$

which is the same as $a_{k\gamma}$ but differ in the sign of $\left(\beta = -\dfrac{u}{c}\right)$.

From (I) and (II) we can obtain Lorentz transformation equations for the co-ordinated and time as follows:

Putting $k = 1$ in (I) and using the matrix form of $a_{k\gamma}$ we have:

$$x'_1 = a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 + a_{14}\ x_4$$

but $a_{11} = \alpha$, $a_{12} = 0$, $a_{13} = 0$, $a_{14} = i\beta\alpha$

$$x_1 = x \ ,\ x_2 = y \ ,\ x_3 = z,\ x_4 = ict$$

$$\therefore x'_1 = \alpha\ x + (i\beta\ \alpha)\ (ict) = \alpha(x - \beta\ ct)$$

$$x' = \alpha(x - Vt) \qquad (1)$$

If we put k = 2 and k = 3 we have:

$$x'_2 = y' = a_{21}\ x_1 + a_{22}\ x_2 + a_{23}\ x_3 + a_{24}\ x_4$$

$$y' = 0 + (1)\ (y) + 0 + 0 = y \qquad (2)$$

$$x'_3 = z' = a_{31}\ x_1 + a_{32}\ x_2 + a_{33}\ x_3 + a_{34}\ x_4$$

$$z' = 0 + 0 + (1)\ (z) + 0 = z \qquad (3)$$

for k = 4:

$$x'_4 = ict' = a_{41}\ x_1 + a_{42}\ x_2 + a_{43}\ x_3 + a_{44}\ x_4$$

$$= (-i\beta\alpha)\ (x) + 0 + 0 + (\alpha)\ (ict)$$

$$ict' = \alpha\ (ic)\left(t - \frac{V}{c^2}x\right)$$

$$\therefore t' = \alpha\left(t - \frac{V}{c^2}x\right) \qquad (4)$$

The transformation law (I) is know as the covariant (or the tonsorial) form of the Lorentz transformation. For any 4-vector A_u this law is hold:

i.e. $\quad A'_k = a_{k\gamma}\ A_\gamma$

and inversely:

$$A_k = a'_{k\gamma}\ A'_\gamma$$

where $a_{k\gamma}$ and $a'_{k\gamma}$ are given by (II) and (IV).

4- The velocity 4-vector:

we define the velocity 4-vectors of a particle in the 4-dimensional space as:

$$U_k = \frac{dx}{dT} \quad , \quad k = 1, 2, 3, 4$$

and $dt = dt\sqrt{1-\beta^2}$ is the proper time:

$$U_k = (U_1, U_2, U_3, U_4) = (\bar{U}, U_4)$$

$$U_1 = \frac{dx_1}{dT} = \frac{dx_1}{dt\sqrt{1-\beta^2}} = \frac{V_x}{\sqrt{1-\beta^2}} = \alpha V_x$$

where $$\alpha = \frac{1}{\sqrt{1-\beta^2}} \quad , \quad V_x = \frac{dx}{dt}$$

$$U_2 = \frac{dx_2}{dT} = \frac{dx_2}{dt\sqrt{1-\beta^2}} = \frac{V_y}{\sqrt{1-\beta^2}} = \alpha V_y$$

$$U_3 = \frac{dx_3}{dT} = \frac{dx_3}{dt\sqrt{1-\beta^2}} = \frac{V_z}{\sqrt{1-\beta^2}} = \alpha V_z$$

$$U_4 = \frac{dx_4}{dT} = \frac{ic\ dt}{dt\sqrt{1-\beta^2}} = \frac{ic}{\sqrt{1-\beta^2}} = \alpha(ic)$$

$$\therefore U_k = \alpha(V_x\ ,\ V_y\ ,\ V_z\ ,\ ic) = \alpha(\bar{V}\ ,\ ic)$$

or $$U_k = \left(\frac{\bar{V}}{\sqrt{1-\beta^2}}\ ,\ \frac{ic}{\sqrt{1-\beta^2}}\right)$$

$\bar{V}$ is the classical 3-dimensional velocity.

The magnitude of the velocity 4-vector is defined as:

$$U_k^2 = \sum_{k=1}^{4} U^2 = U_1^2 + U_2^2 + U_3^2 + U_4^2$$

$$= \alpha^2 (V_x^2 + V_y^2 + V_z^2 - c^2) = \alpha^2 (V^2 - c^2)$$

$$= \frac{V^2 - c^2}{1 - V^2/c^2} = -c^2 = \text{invariant}.$$

The law of transformation of the components U_k:

$$U_k = a_{k\gamma}\, U'_\gamma$$

where:

$$a_{k\gamma} = \begin{bmatrix} \alpha & 0 & 0 & -i\beta\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\alpha & 0 & 0 & \alpha \end{bmatrix}$$

For k = 1:

$$\therefore U_1 = a'_{1\gamma}\; U'_\gamma = a'_{11}\; U'_1 + a'_{14}\; U'_4$$

$$= \alpha\; (U'_1 - i\beta\alpha\; U'_4)$$

$$= \alpha\; (U'_1 - i\beta\; U'_4) \quad \text{or} \quad U_1 = \frac{U'_1 - i\beta\; U'_4}{\sqrt{1-\beta^2}} \quad (1)$$

but $\qquad U_1 = \alpha\; V_x \quad , \quad U'_1 = \alpha\; V'_x \quad , \quad U'_4 = \alpha\; (ic)$

$$\therefore \alpha V_x = \alpha\ [V'_x - (i\beta)\ (ic)] = \alpha\ (V'_x + V) = \frac{V'_x - V}{\sqrt{1-\beta^2}} \qquad (1)'$$

For $k = 2$:

$$U_2 = a'_{2\gamma}\ U'_\gamma = a'_{22}\ U'_2\ U'_2 \qquad (2)$$

$$\therefore \alpha V_y = \alpha V'_y \qquad \text{or} \qquad V_y = V'_y \qquad (2)'$$

For $k = 3$:

We have $U_3 = U'_3$ (3)

Or $V_z = V'_z$ (3)'

For $k = 4$:

$$U_4 = a'_{4\gamma}\ U'_\gamma = a'_{41}\ U'_1 + a'_{44}\ U'_4$$

$$= i\beta\ \alpha\ U'_1 + \alpha\ U'_4 = \alpha\ (U'_4 + i\beta\ U'_1)$$

$$U_4 = \frac{U'_4 + i\beta U'_1}{\sqrt{1-\beta^2}} = \frac{\alpha ic + i\beta\alpha V_2}{\sqrt{1-\beta^2}} = \alpha^2 i\ (c + \beta V_x) \qquad (4)$$

equations (1), (2), (3), (4) are the transformations equation for the component of the velocity 4-vector U_k ($k = 1, 2, 3, 4$).

5- The acceleration 4-vector:

It is defined as:

$$W_k = \frac{dU_k}{dT}, \quad \text{where } dT = dt\ \sqrt{1-\beta^2}$$

Or $W_k = \frac{d^2 x_k}{dT^2}$

$$W_k = (W_1 \ , \ W_2 \ , \ W_3 \ , \ W_4) = (\bar{W} \ , \ W_4)$$

$$W_1 = \frac{dU_1}{dT} = \frac{dU_1}{dt} \cdot \frac{dt}{dT} = \alpha \cdot \frac{d}{dt}(\alpha V_x)$$

Since:

$$\frac{dt}{dT} = \alpha \quad , \quad \alpha = \frac{1}{\sqrt{1-\beta^2}} \quad , \quad U_1 = \alpha V_x$$

Also $W_2 = \frac{dU_2}{dT} = \frac{dU_2}{dt} \cdot \frac{dt}{dT} = \alpha \ \frac{d}{dt}(\alpha V_y)$

$$W_3 = \frac{dU_3}{dT} = \frac{dU_3}{dt} \cdot \frac{dt}{dT} = \alpha \ \frac{d}{dt}(\alpha V_z)$$

$$W_4 = \frac{dU_4}{dT} = \frac{dU_4}{dt} \cdot \frac{dt}{dT} = \alpha \ \frac{d}{dt}(ic \ \alpha) = 0$$

$$W_k = \alpha \left[\frac{d}{dt}(\alpha \bar{V}) \ ; \ \frac{d}{dt}(ic \ \alpha)\right]$$

$$= \frac{1}{\sqrt{1-\beta^2}}\left[\frac{d}{dt}\left(\frac{\bar{V}}{\sqrt{1-\beta^2}}\right) \cdot \frac{d}{dt}\left(\frac{ic}{\sqrt{1-\beta^2}}\right)\right] \cdot$$

Since $U_k^2 = -c^2$, then differentiating w . r . t. T we have:

$$U_k = \frac{dU_k}{dT} = U_k \ . \ W_k = 0$$

This means that the 4-vectors U_k and W_k are orthogonal to each other in the 4-dimensionsl spaces.

6- Geometrical representation of special relativity:

a) Introduction:

The Lorentz transformations from a system S to anther one S' are given as:

$$x' = \alpha\ (V)[x - Vt]$$

$$y' = y$$

$$z' = z$$

$$t' = \alpha\ (V)\left[t - \frac{V}{c^2}x\right]$$

making use of the new coordinates:

$$x_1 = x \quad , \quad x_2 = y \quad , \quad x_3 = z\ , \quad x_4 = ict$$

and taking $V = -ic\tan\theta$ or $\beta = \frac{V}{c} = -i\tan\theta$ Lorentz transformation then take the form:

$$x'_1 = x_1\cos\theta + x_4\sin\theta$$

$$x_2 = x'_2$$

$$x_3 = x'_3$$

$$x'_4 = x_4\cos\theta - x_1\sin\theta$$

These show that in order to transform observations of the system S to that of S' one has to use the sew coordinates x_1, x_2, x_3, x_4 and the Lorentz transformation is not more than a rotation of the two axes x_1 and x_4 through an imaginary angle $\theta = \tan^{-1}\left(\frac{iV}{c}\right)$.

Such four dimensional space is called Minkowski's four-dimensional space-time.

Restricting ourselves to consider any event as described in system S by two coordinates x_1 and x_4 such that:

x_1 defines its position.

x_4 defines its time.

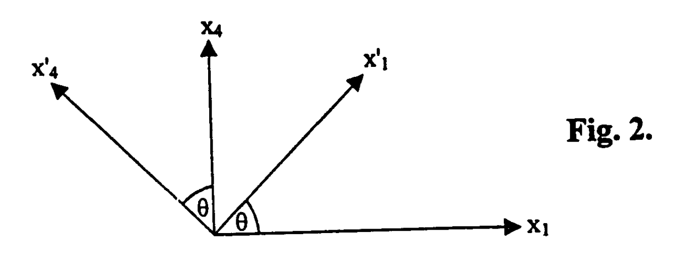


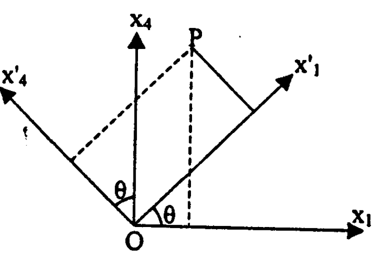
Fig. 2.

The same event is to be described in system S' by two coordinates x'_1 and x'_4 obtained from x_1 and x_4 by redefine the axes in the manner described to redefine the corresponding position and time. This means that any concerned event is completely represented by a point in the 4-diBensional space by

a set of 4-coordinatea (x_1 , x_2 , x_3 , x_4), this representation is done in the following manner.

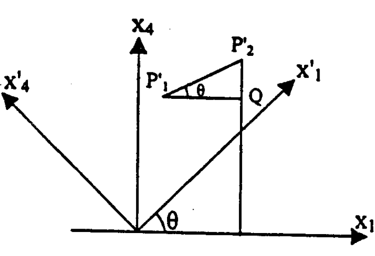
b) Representation of events in Minkowski space:

(i) An event:



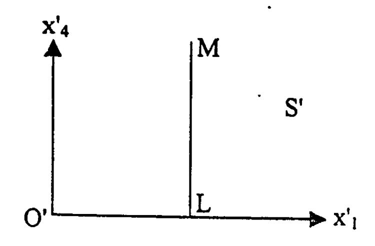
An event is represented in the 4-dlmonsional space by a point $P \equiv (x_1 , x_4)$ denoting its position and time in S and $P \equiv (x'_1 , x'_4)$ denoting corresp-onding position and time in S'.

(ii) Simultaneity of events:



Two simultaneous events P'_1 and P'_2 in a system S' show a time difference, in the system S as corresponding to P'_2 Q.

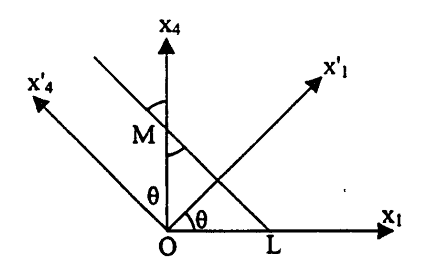
(iii) A particle at rest:



A particle at rest in a system S' will be represented by a set of points all having the same x'_1 (since it is at rest) and therefore the will lie on a straight line // to the x'_4 axis, such as LM. This line in called the world, line of the particle.

(iv) A moving particle:

Consider a particle in the system S to move along its x-direction with some uniform velocity V = kv, where k is Bone factor.



If the particle starts its motion from a point x_o then, since the particles moves on a straight line, we can write:

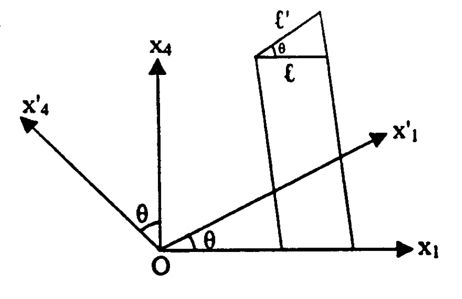
$$
\begin{aligned}
x_1 &= x_o + V\,t \\
&= x_o + kvt \\
&= x_o + k\ (-ic\tan\theta)\frac{x_4}{ic} \\
&= x_o - k\tan\theta\ .\ x_4
\end{aligned}
$$

For k = 1, the concerned particle has S' frame as its proper frame. In such a case the world line of the point is a straight line LM // to ox'4. The world line of a point moving in any general manner will be a curve.

c) Fitzgerald length contraction:

Consider a system being at rest with respect to S' frame. The distance $\ell' = \ell_0$ measuring the length between two of its material points.

The corresponding separation distance in the S frame will then be ℓ such that:

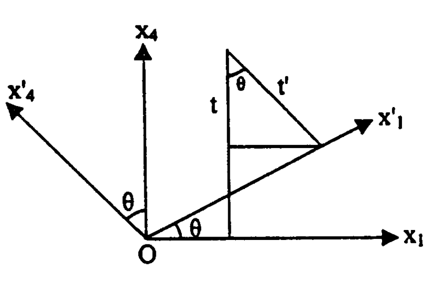


$$\ell' = \ell \cos\theta = \ell \ . \ \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

The definition of the length is, to be consistent with simultaneous measurement of the end points from each of the frames. Notice that due to the figure $\ell > \ell'$ in view of being Imaginary one.

d) The time dilation:

Consider a tine duration of some event having S' as its proper trams being measured as $t' = t_o$. A corresponding time interval due to the S will be such that:



$$t = t' \ . \ \cos\theta = \frac{t'}{\sqrt{1-\frac{V^2}{c^2}}}$$

7- The momentum 4-vector:

The momentum of a moving particle can be represented by 4-dimensional vector as; $P_k = m_o U_k$, which is known as the momentum 4-vector, the components of this vector are given by:

$$P_1 = P_x = \frac{m_o V_x}{\sqrt{1-\beta^2}} \;,\; P_2 = P_y = \frac{m_o V_y}{\sqrt{1-\beta^2}} \;,\; P_3 = P_z = \frac{m_o V_z}{\sqrt{1-\beta^2}}$$

$$P_4 = m_o U_4 = \frac{dx_4}{dT} = \frac{ic\ m_o}{\sqrt{1-\beta^2}}, \quad \text{where} \quad \beta = \frac{V}{c}$$

$$\therefore P_k = \left(\frac{m_o \vec{V}}{\sqrt{1-\mu^2}} = \frac{i\ m_o c}{\sqrt{1-\beta^2}} \right)$$

$$= m_o \alpha\ (\vec{V}\ .\ ic) = m\ (\vec{V}\ .\ ic) = (\vec{P}\ .\ i\ m\ c)$$

$$m = m_o \alpha = \frac{m_o}{\sqrt{1-\beta^2}}$$

$\vec{P}$ is the classical 3-dimensional momentum in the no relativistic case $V << 0.$, $\alpha \to 1.$

$$\therefore P_x = m_o v_x \;,\; P_y = m_o v_y \;,\; P_z = m_o v_z \;\text{ or }\; \vec{P} = m_o \vec{v}$$

Also, since the energy of the particle is:

$$E = mc^2 = \frac{m_o c^2}{\sqrt{1-\beta^2}}$$

and $$P_4 = i\ m\ c = \frac{i\ m_o c}{\sqrt{1-\beta^2}}$$

or $$P_4 = i\ m\ c = \frac{i}{c} mc^2 = \frac{i}{c} E$$

Thus the momentum 4-vctor is given by:

$$P_k = (m\ \vec{V}\ ,\ \frac{i}{c} E) = (\vec{P}\ ,\ \frac{iE}{c})$$

Theorem: We can show that $\sum_{k=1}^{4} (P_k)^2$ invariant, since:

$$\sum_{u=1}^{4} (P_u)^2 = P_k P_k = P_1 P_1 + P_2 P_2 + P_3 P_3 + P_4 P_4$$

$$= P_x^2 + P_y^2 + P_z^2 + \left(\frac{iE}{c}\right)\left(\frac{iE}{c}\right)$$

$$= (\vec{P})^2 - \frac{E^2}{c^2} = (\vec{P})^2 - \frac{(mc^2)^2}{c^2} = P^2 - m^2c^2$$

$$= \text{invariant}$$

This means that:

$$P^2 - m^2c^2 = P'^2 - m'^2\ c^2 = \text{invariant}$$

Also since $P_k = m_o U_k$.

$$\therefore P_u\ P_u = \sum_{k=1}^{4} (P_k)^2 = m_o^2 \sum_{k=1}^{4} (U_k)^2$$

but $$\sum_{k=1}^{4} (U_k)^2 = U_k\ U_k = -c^2$$

$$\therefore P_k\ P_k = m_o^2 c^2 = \text{invariant}$$

8- The transformation equations for momentum:

We have seen that if:

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 = \text{invariant}$$

Then the trans formation from the system S to S' is a Lorentz one. The transformation equations for the co-ordinates. and time are:

$$x' = \alpha\ (x - vt)\ \ ,\ \ y' = y\ \ ,\ \ z' = z\ \ ,\ \ t' = \alpha\left(t - \frac{v}{c^2}x\right)$$

Similarly since we have $P^2 - m^2c^2 = P'^2 - m'^2\,c^2 = \text{invariant}$.

Than the transformation equations for the momentum P and mass m are:

$$P'_x = \alpha\ (P_x - vm)\ ,\ P'_y = P_y\ \ ,\ \ P'_z = P_z\ \ ,\ \ m' = \alpha\left(m - \frac{v}{c^2}P_x\right)$$

replacing m by $\frac{E}{c^2}$ we can get the transformation equation between energy E and momentum P as:

$$E' = \alpha\ (E - vP_x)\ ,\ P'_y = P_y\ \ ,\ P'_z = P_z\ \ ,\ P'_x = \alpha\left(P_x - \frac{v}{c^2}E\right)$$

Where:

$$\alpha = \frac{1}{\sqrt{1-\beta^2}}$$

Noting that:

From the relation between energy and momentum:

$$E^2 = P^2c^2 + m_o^2c^4$$

$$E^2 - P^2c^2 = m_o^2c^4 = \text{invariant}$$

Or $$E^2 - P^2c^2 = E'^2 - P'^2 c^2 = \text{invariant}$$

Note also that: We can get the transformation equation of energy and momentum as follows:

Since $$P'_k = \sum_{k=1}^{4} a_{k\gamma} P_k$$

Where:

$$a_{k\gamma} = \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{-i\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{bmatrix}$$

$$\therefore P'_x = P'_1 = a_{11} P_1 + a_{14} P_4$$

$$= \frac{P_1 + i\beta P_4}{\sqrt{1-\beta^2}} = \alpha\left[P_x + i\frac{v}{c} , \frac{iE}{c}\right]$$

$$= \alpha\left(P_x - \frac{v}{c^2} E\right) \quad \text{and } P'_y = P_y \ , \ P'_z = P_z$$

$$P'_4 = \frac{E'}{c} = a_{41} P_I + a_{44} P_4 = \frac{-i\beta P + P}{\sqrt{1-\beta^2}}$$

$$= \alpha\left(-i\frac{v}{c} P_x + \frac{i}{c} E\right) = \alpha \frac{i}{c}(-VP_x + E)$$

$$\therefore E' = (E - VP_x)$$

9- The force 4-vector (Eqn of motion in relativistic theory):

The generalization of Newton's second law of classical dynamics in S.R is:

$$\frac{dP_k}{dT} = \frac{d}{dT}(m_o U_k) = F_k \qquad \text{(I)}$$

Where P_k is Know as the force 4-vector or Minkowski-force.

Since $\quad P_k = (\vec{P} \ , \ imc) \quad , \quad U_k = \alpha(\vec{V} \ , \ ic)$

And since $dT = \sqrt{1-\beta^2}\ dt$ or $\dfrac{d}{dT} = \dfrac{1}{\sqrt{1-\beta^2}} \dfrac{d}{dt} = \alpha \dfrac{d}{dt}$

Thus the components of the force 4-vector are:

$$P_k = \alpha \frac{dP_u}{dt} = \left(\frac{d\vec{P}}{dt} \ , \ ic\frac{dm}{dt}\right) = \alpha\left(\vec{f} \ , \ ic\frac{dm}{dt}\right)\left(\vec{P} \ , \ ic\alpha\frac{dm}{dt}\right)$$

where $\vec{f}$ is the classical 3-dimensional force, $\alpha\vec{f} = \vec{F}$.

Theorem: The force 4-vector F_k act along the principle normal to the path of the particle in the 4-dimensional Minkowski space.

i.e. $$(U_k \ , \ F_k) = 0$$

or $$\sum_{k=1}^{4} U_k F_k = 0 \qquad \text{or} \qquad U_k F_k = 0$$

which means that:

The inner product of the two 4-vectors U_k and F_k vanishes.

Since $$\sum_{k=1}^{4} U_k U_k = -c^2 = U_k^2$$

Differentiate w . r . t. T we have:

$$2U_k \frac{dU_k}{dT} = 0 \quad , \qquad \therefore U_k m_o \frac{dU_k}{dT} = 0$$

$$\therefore U_k F_k = 0 \qquad \text{or} \qquad (U_k \ , \ F_k) = 0$$

Example:

Using the facts that $(U_k \ , \ F_k) = 0$ obtain the relation between mass and energy in S.R.

Solution:

Since $U_k = \alpha(\vec{V} \ , \ ic)$ and $F_k = \alpha\left(\vec{f} \ , \ ic\frac{dm}{dt}\right)$

$$(U_k \ , \ F_k) = 0$$

Or $$\sum_{k=1}^{4} U_k F_k = U_1 F_1 + U_2 F_2 + U_3 F_3 + U_4 F_4 = 0$$

$$\therefore \alpha^2\left(\bar{f}\,.\,\bar{V} - c^2\frac{dm}{dt}\right) = 0$$

$$\therefore \bar{f}\,.\,\bar{V} - c^2\frac{dm}{dt}$$

$$\therefore \bar{f}\,.\,\frac{\overline{dr}}{dt} = \frac{dW}{dt} = \frac{dT}{dt} = c^2\frac{dm}{dt}$$

where $dW = \bar{f}\,.\,\overline{dr}$, the work done by the force $\bar{F}$.

$$\therefore dT = c^2 dm \qquad \therefore T = c^2 m + \text{constant}$$

To find the constant:

The kinetic energy $T = 0$ when $m = m_o$.

$$\therefore \text{constant} = -c^2 m_o$$

$$\therefore T = c^2(m - m_o) = E - E_o$$

where E_o is the rest energy.

Thus the total energy $E = c^2 m$ which is the relation between energy and mass.

Note that: The eqn of motion in relativistic theory (I) can be written as follows:

$$\frac{dP_x}{dT} = \alpha\frac{dP_x}{dT} = \alpha\frac{d}{dT}\,(m_o\alpha V_x) = F_x$$

Or $$\frac{d}{dT}(m_o \alpha V_x) = \frac{F_x}{\alpha} = F_x\sqrt{1-\beta^2} = f_x$$

Where f_x is the ordinary force of classical mechanics.

The other components:

$$\frac{d}{dt}(m_o \alpha V_y) = \frac{F_y}{\alpha} = f_y$$

$$\frac{d}{dt}(m_o \alpha V_z) = \frac{F_z}{\alpha} = f_z$$

Or $$\frac{d}{dt}(m_o \alpha \vec{V}) = \frac{\vec{F}}{\alpha} = \vec{f}, \text{ where } \vec{F} = \alpha \vec{f}$$

Also the Fourth component is:

$$\frac{d}{dt}(ic\ m_o \alpha) = \frac{F_4}{\alpha} = f_4$$

Where $m_o \alpha = m$., $\alpha = \frac{1}{\sqrt{1-\beta^2}}$

Finally the equation of motion can be written in the form:

$$\frac{d}{dt}(m_o \alpha U_k) = f_k \quad \text{or} \quad \frac{d}{dt}(m\ U_k) = f_k$$

Note also that:

Since $$F_k = \alpha\left(\vec{f}\ ,\ ic\frac{dm}{dt}\right)$$

$$m = \frac{E}{c^2}\ ,\ F_k \equiv \alpha\left(\vec{f}\ ,\ \frac{i}{c}\frac{dE}{dt}\right) \equiv \left(\vec{F}\ ,\ \frac{i}{c}\alpha\frac{dE}{dt}\right)$$

10- The particles of zero rest mass:

We can easily show that any particle move with the velocity of light C (such as photon, neutrinos,....), will have a zero rest mass.

Since $\quad P^2 - P^2c^2 = -m_o^2c^2$

$$\therefore (mc)^2 - m^2c^2 + m_o^2c^2 = 0$$

But $\quad c \neq 0 \qquad m_o = 0$

For this reason such particles are called particles of zero rest mass.

These particles are always associated with a certain type of waves of frequency γ and wave length λ; the energy of such particles. $E = h\gamma$, $\gamma = \frac{c}{\lambda}$, h is known as Plank's constant.

The momentum 4-vector of the particle of zero rest mass is:

$$P_k = (\bar{P}\ ,\ imc) = (mc\hat{P}\ ,\ imc) = \alpha\left(\frac{E}{c}\hat{P}\ ,\ i\frac{E}{c}\right) = \left(\frac{h\gamma}{c}\hat{P}\ ,\ i\frac{h\gamma}{c}\right)$$

where $\hat{P}$ is a unit vector in the direction of P, and the magnitude of classical (3-dimensional) momentum is thus equal to $\frac{h\,\gamma}{c}$. Also since $(P_k)^2 = P_k\ P_k = -m_o^2c^2$.

Thus for the particle of zero rest mass ($m_o = 0$).

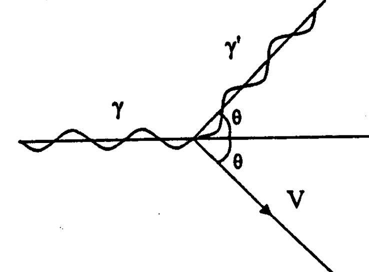
$$\therefore (P_k)^2 = 0$$

11- Some applications in physics:

Collision of relativistic particles:

Example: Compton effect.

A photon (with rest mass = 0) collides elastically with can electron. If the electron is at rest before collision and if γ and γ' are the frequencies of the photon radiation before and after collision and θ , $\varnothing$ are the angles of scattering of photon and electron respectively, thus the momentum 4-vectora will be:



For the photon:

$$P_{\gamma} = \left(\frac{h\gamma}{c}\hat{p} \; , \; i\frac{h\gamma}{c}\right) \text{ before collision.}$$

$$P'_{\gamma} = \left(\frac{h\gamma'}{c}\hat{p}' \; , \; i\frac{h\gamma'}{c}\right) \text{ after collision.}$$

For the electron:

$$P_e = \left(0 \; , \; im_o c\right) \text{ before collision.}$$

$$P'_e = \left(m\vec{V} \; , \; imc\right) \text{ after collision.}$$

The laws of conservation of momentum and energy can written as:

$$P_{\gamma} + P_e = P'_{\gamma} + P'_e$$

$$\therefore P'_e = P_\gamma + P_e - P'_\gamma$$

Consider the inner product of each side with itself:

$$\therefore (P'_e)^2 = (P_\gamma^2) + (P_e)^2 + (P'_\gamma)^2 + 2\,(P_\gamma , P_e)$$

$$- 2\,(P_\gamma , P'_\gamma) - 2\,(P_e , P'_\gamma)$$

But $\quad (P_\gamma)^2 = 0 \quad , \quad (P'_\gamma)^2 = 0 \quad$ (for photons)

And:

$$(P'_e)^2 = (P_e)^2 = -m_o^2c^2 \qquad \text{(since it is invariant)}$$

where m_o is mass of the electron.

$$\therefore (P_\gamma , P_e) - (P_e , P'_\gamma) = (P_\gamma , P'_\gamma)$$

$$\therefore -h\ m_o\gamma + h\ m_o\gamma' = \frac{h\gamma'}{c} \cdot \frac{h\gamma'}{c}\cos\theta - \frac{h^2\gamma\gamma'}{c^2} \quad , \quad P \cdot \hat{P}' = 1$$

$$m_o(\gamma - \gamma') = \frac{h\ \gamma\gamma'}{c^2}(1 - \cos\theta)$$

$$\frac{\gamma - \gamma'}{\gamma\gamma'} = \frac{h}{m_o c^2}(1 - \cos\theta)$$

$$\frac{1}{\gamma'} - \frac{1}{\gamma} = \frac{h}{m_o c^2}(1 - \cos\theta)$$

$$\frac{c}{\gamma'} - \frac{c}{\gamma} = \frac{h}{m_o c}(1 - \cos\theta)$$

$$\therefore \lambda' - \lambda = \frac{2h}{m_o c}\sin^2\frac{\theta}{2}$$

which the well know formula.

Solved problems

Example (1):

Two events $P_1 : x_1 = 10$ m , $t_1 = 2 \times 10$ sec.

And $P_2 : x_2 = 50$ m , $t_2 = 3 \times 10^{-7}$ sec.

Obtain the distance between the two events and the time difference bet. Them as measured in the frame S' moving by the velocity V = 0.6 C.

$$V = 0.6\,C \qquad \therefore \alpha = \frac{5}{4}$$

$$x_1 = 10\ \text{m} \quad , \quad t_1 = 2 \times 10^{-7}\ \text{sec.}$$

$$x_2 = 50\ \text{m} \quad , \quad t_1 = 3 \times 10^{-7}\ \text{sec.}$$

The distance bet. The events as measured in S' is given by:

$$x'_2 - x'_1 = \alpha\,[\,(x_2 - x_1) - V\,(t_2 - t_1)]$$

$$= \frac{5}{4}\left[(50-10) - \frac{3}{5}(3\times10^8)10^{-7}\right]$$

$$= \frac{5}{4}[40-18] = \frac{5.22}{4} = \frac{55}{2} = 27.5\ \text{m}$$

The time difference is given by:

$$t'_2 - t'_1 = \alpha\,[\,(t_2 - t_1) - \frac{V}{c^2}\,(x_2 - x_1)]$$

$$= \frac{5}{4}\left[10^{-7} - \frac{3}{5}\ \frac{(50-10)}{3\times10^8}\right]$$

$$= \frac{5}{4}\left[10^{-7} - \frac{3-40}{5\times3\times10^{8}}\right]$$

$$= \frac{5}{4}\left[10^{-7} - 0.8\times10^{-7}\right]$$

$$= \frac{5}{4} \cdot 10^{-7}(1-0.8) = \frac{5}{4}\times\frac{2}{10}\times10^{-7}$$

$$= 2.5\times10^{-8}\ \text{sec}.$$

$$= 0.25\times10^{-7}\ \text{sec}.$$

Example (2):

Two events occur at the same place in a certain inertial frame S and are separated by a time interval of 4 sec. What the spatial separation between these two events in an inertial frame in which the events are separated by a time interval 6 sec.

Solutions:

$$x_2 - x_1 = 0 \quad , \quad t_2 - t_1 = 4\ \text{sec}.$$

$$x'_2 : x'_1 = ? \quad , \quad t'_2 - t'_1 = 6\ \text{sec}.$$

$$x'_2 - x'_1 = \alpha\,[\,(x_2 - x_1) - V\,(t_2 - t_1)] \qquad (1)$$

$$t'_2 - t'_1 = \alpha\,[\,(t_2 - t_1) - \frac{V}{c^2}\,(x_2 - x_1)] \qquad (2)$$

$$(2) \rightarrow \qquad 6 = 4\,\alpha \qquad \therefore \alpha = \frac{6}{4} = \frac{3}{2}$$

$$\therefore \alpha = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \therefore \frac{9}{4} = \frac{1}{1-\frac{v^2}{c^2}}$$

$$\therefore 9 - 91 - \frac{v^2}{c^2} = 4 \qquad \therefore 9\frac{v^2}{c^2} = 5 \qquad \therefore V = \frac{\sqrt{5}}{3}C$$

$$\therefore x'_2 - x'_1 = \frac{3}{2}\left[-\frac{\sqrt{5}}{3}C \ . \ 4\right] = 2\sqrt{5}\ C$$

$$= 2\sqrt{5}\ .\ 3\times10^5 \text{km} = 6\sqrt{5}\times10^5 \text{km}$$

Example (3):

Prove, with the usual notations, that:

$$1-\frac{v^2}{c^2} = \left(1-\frac{vu_x}{c^2}\right)\left(1+\frac{vu'_x}{c^2}\right)$$

$$c^2\left(c^2+V^2\right) = \left(c^2+Vu_x\right)\left(c^2+Vu'_x\right)$$

Solution:

Since
$$dt' = \frac{dt - \frac{v}{c^2}dx}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$dt = \frac{dt' - \frac{v}{c^2}dx'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore dt \,.\, dt' = \frac{dt'\left(1+\frac{v}{c^2}\frac{dx'}{dt'}\right) . \, dt\left(1-\frac{v}{c^2}\frac{dx}{dt}\right)}{\left(1-\frac{v^2}{c^2}\right)}$$

$$= \frac{dt \,.\, dt'}{\left(1-\frac{v^2}{c^2}\right)}\left(1+\frac{v}{c^2}u'_x\right) . \left(1-\frac{v}{c^2}u_x\right)$$

$$1-\frac{v^2}{c^2} = \left(1+\frac{v}{c^2}u_x\right)\left(1-\frac{v}{c^2}u'_x\right)$$

Example (4):

A particle P has the motion $x = \frac{c}{g}\sqrt{c^2 + g^2t^2}$ along the x-axis of a forenty frame S and a second particle Q with the path $x = \frac{1}{2}ct + \frac{c^2}{g}$ along the x-axis just passes P at t = 0. Show that according to a clock moving with Q the two particles meet again after a time $\frac{2c}{g\sqrt{3}}$.

$$V_Q = V = \frac{c}{2}$$

$$x_P = \frac{c}{g}\sqrt{c^2 + g^2t^2}$$

$$x_Q = \frac{1}{2}ct + \frac{c^2}{g}$$

A (x_1, t_1) B (x_2, t_2)

O

$\frac{c^2}{g}$

$$\therefore \frac{c^2}{g^2}\left(c^2+g^2t^2\right)=\left(\frac{ct}{2}+\frac{c^2}{g}\right)^2$$

$$\therefore t_1=0 \quad , \quad t_2=\frac{4}{3}\frac{c}{g}$$

then the two particles meet again at the point B, where:

$$t_B=\frac{1}{2}c\ .\ \frac{4}{3}\frac{c}{g}+\frac{c^2}{g}=\frac{5}{3}\frac{c^2}{g}$$

$$B(x\ ,\ t)=B\left(\frac{5c^2}{3\,g}\ ,\ \frac{4}{3}\frac{c}{g}\right)$$

$$x_2=\frac{2}{3}\frac{c^2}{g}$$

$$t_2=\frac{4}{3}\frac{c}{g}$$

O

$$t'_2=\alpha(t_2-\frac{v}{c^2}x^2)$$

$$V=\frac{c}{2}$$

O

$$\alpha=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{2}{\sqrt{3}}$$

$$=\frac{2}{\sqrt{3}}\left(\frac{4}{3}\frac{c}{g}-\frac{c}{2c}\ .\ \frac{2}{3}\frac{c^2}{g}\right)$$

$$=\frac{2}{\sqrt{3}}\left(\frac{4}{3}\frac{c}{g}-\frac{1}{3}\frac{c}{g}\right)$$

$$t'_2 = \frac{2}{\sqrt{3}} \frac{c}{g}$$

The two particles meet after time $\frac{2c}{\sqrt{3}g}$ according to a clock moving with the particle Q.

Example (5):

Prove that the max.vel. with which a particle can move after it has been accelerated with a constant force in a straight line is equal to the velocity of light. Deduce the equation of the linear displacement as function of the time and the force.

Solution:

The force is given by:

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(\overline{mV}) = \frac{d}{dt}\left(\frac{m_o V}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

Since the motion takes place in a straight line we get:

$$F = \frac{d}{dt}\left(\frac{m_o V}{\sqrt{1 - \frac{v^2}{c^2}}}\right) \quad (1)$$

$$\therefore F\ dt = d\left(\frac{m_o V}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

Integrating we get:

$$\int_0^t F\ dt = \int_0^V d\frac{m_o V}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore F\ t = \frac{m_o V}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \frac{F\ t}{m_o c} = \frac{V/c}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \left(\frac{F\ t}{m_o c}\right)^2 = \frac{\left(\frac{V}{c}\right)^2}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)}$$

$$\therefore \left(\frac{V}{c}\right)^2 = \left(\frac{F\ t}{m_o c}\right)^2\left(1-\frac{v^2}{c^2}\right) = \left(\frac{F\ t}{m_o c}\right)^2 - \frac{v^2}{c^2}\left(\frac{F\ t}{m_o c}\right)^2$$

But $$\left(\frac{F\ t}{m_o c}\right)^2 = \left(\frac{V}{c}\right)^2\left[1+\left(\frac{F\ t}{m_o c}\right)^2\right]$$

$$\therefore \frac{V}{C} = \frac{\dfrac{F\,t}{m_o c}}{\sqrt{1 + \left(\dfrac{F\,t}{m_o c}\right)^2}} \qquad (2)$$

It is seen, from the last relation, that $\frac{V}{C} = \frac{F\,t}{m_o c}$, when t is too small, $\left(\frac{F\,t}{m_o c}\right)^2 = 0$.

In this case we have:

i.e. $$V = \left(\frac{F}{m_o}\right) t \qquad (3)$$

Which is the classical limit for velocity since $\frac{F}{m_o}$ represents the acceleration of the particle.

Also $$\frac{V}{C} = \frac{\dfrac{F\,t}{m_o c}}{\dfrac{F\,t}{m_o c}} = 1 \text{ when t is to large,}$$

i.e. $$\sqrt{1 + \left(\frac{F\,t}{m_o c}\right)^2} = \frac{F\,t}{m_o c}$$

In this case we get:

$$V = c$$

To obtain the displacement x we substitute:

$$V = \frac{dx}{dt} \qquad \text{in (2)},$$

$$\therefore V = \frac{dx}{dt} = c\frac{\left(\frac{F}{m_o c}\right)t}{\sqrt{1+\left(\frac{F\,t}{m_o c}\right)^2}}$$

$$\therefore \int_0^x dx = \int_0^t c\frac{\left(\frac{F}{m_o c}\right)t}{\sqrt{1+\left(\frac{F\,t}{m_o c}\right)^2}}dt$$

$$\therefore x = \frac{m_o c^2}{F}\left[\sqrt{1+\left(\frac{F\,t}{m_o c}\right)^2}\right]_o$$

$$\therefore x = \frac{m_o c^2}{F}\left[\sqrt{1+\left(\frac{F\,t}{m_o c}\right)^2}-1\right] \qquad (4)$$

It is seen that:

$x = \frac{m_o c^2}{F}\left[\frac{1}{2}\left(\frac{F}{m_o c}\right)^2\right] = \frac{1}{2}\frac{F\,t}{m_o}$, when t is too small, which is the class ical limit for displacement.

Also $\qquad x = \frac{m_o c^2}{F}\left[\frac{F\,t}{m_o c} - 1\right] = ct - \frac{m_o c^2}{F}$, when t is to large.

PROBLEMS

1. Two events P and L occur simultaneously in a frame S and 3×10^5 km apart on its x-axis. Find the velocities (as a fraction of C) of the frames, moving parallel to the x-axis of S in which P precedes L by 10 sac.

2. Two events occur at the same time in an inertial frame S and are separated by a distance of 1 km along the x-axis. What is the time difference between these two events as measured in a frame S' in which their spatial separation is measured as 2 km.

3. Particle A is located initially at the origin and moves with a constant velocity $V_x = 0.9c$ along the x-axis. Particle B strictly B shafts at x = 1 and moves with a velocity $V_x = -0.9c$ along the same axis. What is the closure rate of the two particles according to an observer in I; that is, at what rats is the distance between the two decreasing? What is the velocity of B, measured in a frame I' translating with A ?

4. The origins O and O' of the inertial frames I and I' concede at t = t' = 0. I' translates with a constant velocity V relative to I in the positive x direction. At t = 0 a photon leaves the common origin and moves with velocity c, hitting a wall at x = 1 which is fixed in I.

(a) Find the values of x' and t' for the event that the photon hits the wall.

(b) What is the location of the wall relative to O expressed us a function of t'.

5. A relativistic train moves with velocity $V\underline{i}$, along a straight track y = h in frame I. When the front end of the train crosses the y-axis, a relativistic bullet is fired from the origin O with velocity $V = v\underline{j}$.

(a) How far behind the front end of the brain does the bullet strike, according to an observer in frame I?

(b) How far behind its front end does the bullet strike, according to an observer on the train?

(c) What are the x' and y' velocity components of the bullet relative to the train?

6. The inertial frame I' moves relative to 1 with the constant velocity V in the x direction. Find the speed relative to I of a particle moving along the common x axis such that is values of t and t' are equal at all times. Solve for the proper time τ as a function of t.

7. Suppose the inertial frame I_1, translates with velocity v_1, relative to I_o in the direction of the common x axis, Similarly, I_2 translates with velocity v_2 relative to I_1, and the process continues in the same fashion for frames I_3, and I_4. Finally, a particle translates with velocity v_5 relative to I_4. Assuming that $v_1 = v_2 = v_3 = v_4 = v_5 = \frac{1}{2}c$, what is the velocity pf the particle relative to I_o.

8. An inertial frame I' has a constant velocity V In the common x direction relative to the frame I. Suppose a particle has the velocity component (V'_x , V'_y) relative to I' and the corresponding acceleration components (a'_x , a'_y), Show that the acceleration components of the particle relative to the frame I are:

$$a_x = \frac{(1-B^2)^{3/2}}{(1+B'_{V_x}\, c)^3} \quad , \quad a_y = \frac{1-B^2}{(1+BV'_x/c)^3}\left(a'_y - \frac{B\,V'_y\,c}{1+BV_x/c}a'_x\right)$$

where $\beta = V/c$.

9. Prove that the expression for the kinetic energy In relativistic mechanics tends to the known classical- mechanical expression when the particle velocity v is more less than then velocity of light, that Is $v \ll c$.

10. Calculate the generalized length for a straight rod moving in the direction of its Length with a velocity, which makes its relative mass twice its rest mass. (Hint: The generalized length L is defined, as before, by

$L = L_o\sqrt{1-\frac{v^2}{c^2}}$, where L_o is the proper length (or the rest length), and v is the velocity of the rod).

11. A charged π- mason of rest mass m_π is at rest in an inertial frame of references, it be into a u- meson of rest mass m_u and a neutrino (of zero rest mass). Prove that the energies of the 2-particles are:

$$E_u = c^2\left(\frac{m_\pi^2 + m_u^2}{2m}\right) \quad , \quad E_y = c^2\left(\frac{m_\pi^2 - m_u^2}{2m_\pi}\right)$$

12. A neutral π-meson of rest mass m_o moving with velocity V and decays into 2-photons with frequencies γ_1 , $\gamma_2$inclined at angles θ , Ø with the initial direction of the meson, show that:

$$\sin\left(\frac{\theta+\phi}{2}\right) = \frac{m_o c^2}{2h}\left(\frac{1}{\sqrt{\gamma_1\gamma_2}}\right)$$

13. A particle of rest mass M_o moving with velocity V is split into 2 similar particles each of rest mass m_o moving with velocity u at an angle θ with the direction of the Initial particle, prove that:

$$\cos\theta = \frac{c^2}{2m_o v\, u}\left[2m_o - \frac{M_o}{\alpha_1\alpha_2}\right]$$

where $\alpha_1 = \dfrac{1}{\sqrt{1 - v^2/c^2}}$, $\alpha_2 = \dfrac{1}{\sqrt{1 - u^2/c^2}}$

14. A particle of rest mass M is at rest, then decays into 2-particles of rest masses m_1 , m_2 moving with velocities v_1 , v_2, show that:

$$V_1 = c\sqrt{1 - \frac{4M^2 m_1^2}{\left(M^2 + m_1^2 - m_2^2\right)^2}} \quad , \quad V_2 = c\sqrt{1 - \frac{4M^2 m_2^2}{\left(M^2 + m_2^2 - m_1^2\right)^2}}$$

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Part II

Special theory of relativity

Applications



1-Glilean transformations:

(1.1) Events and coordinates.

We begin by considering the concept of a physical event. The event might be the striking of a tree by a lightning bolt or the collision of two particles, and happens at a point in space and at an instant in time. The particular event is specified by an observer by assigning to it four coordinates:

The three positions coordinates x, y, z that measure the distance from the origin of a coordinate system where the observer is located, and the time coordinate t that the observer records with his clock.

Consider now two observers, o and o', where o' travois with a constant velocity v with respect to o along their common $x - x'$ axis fig. (1).

Both observers are equipped with meter sticks and clocks so that they can measure coordinates of events. Further, suppose both observers adjust their clocks so that when they pass each other at $x = x' = 0$, the clocks read $t = t' = 0$. Any given event p will have eight numbers associated with it, the four coordinates (x, y, z, t) assigned by o and the four coordinates (x', y', z', t') assigned (to the same event) by o'.

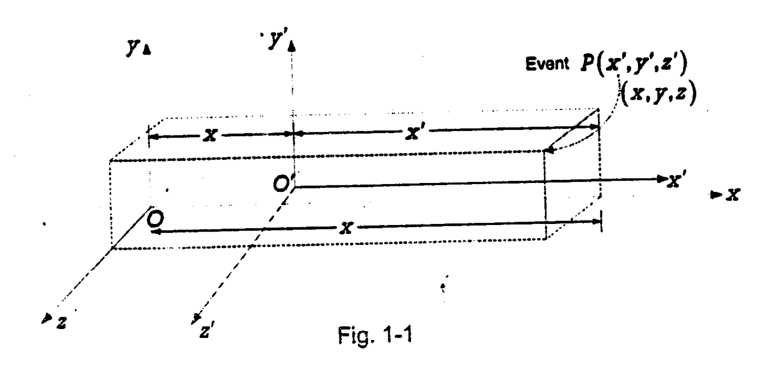


Fig. 1-1

(1.2) Galilean coordinates transformations:

The relationship between the measurements (x, y, z, t) of O and the measurements (x', y', z', t') of O' for a particular event is obtained by examining fig (1):

$$x' = x - vt \qquad y' = y \qquad z' = z$$

In addition, in classical physics it implicitly assumed that

$$t' = t$$

These four equations are called the Galilean coordinate transformations.

(1.3) Galilean velocity transformations :

In addition to the coordinates of an event. The velocity of a particle is of interest. Observers O and O' will describe the particles velocity by assigning three components to it

with (u_x, u_y, u_z) being the velocity components as measured by O , and (u'_x, u'_y, u'_z) being the velocity components as measured

The relationship between (u_x, u_y, u_z) and (u'_x, u'_y, u'_z) is obtained from the time differentiation of the Galilean coordinate transformations. Thus. From $x' = x - vt$.

$$u'_x = \frac{d'x'}{d't} = \frac{d}{dt}(x - vt)\frac{dt}{d't} = \left(\frac{dx}{dt} - v\right)(1) = u_x - v$$

Altogether the Galilean velocity transformations are

$$u'_x = u_x - v \qquad u'_y = u_y \qquad u'_z = u_z$$

(1.4) Galilean acceleration transformations:

The acceleration of a particle, is the time derive of its velocity, i.e. $a_x = du_x / dt$,etc. to find the Galilean acceleration transformations we differentiate the velocity transformations and use the facts that $t' = t$ and v =constant to obtain.

$$a'_x = a_x \qquad a'_y = a_y \qquad a'_z = a_z$$

Thus the measured acceleration components are the same for all observers moving with uniform relative velocity.

(1.5) Invariance of an equation:

By invariance of an equation it is meant that the equation will have the same form when determined by two

observers. In classical theory it is assumed that space and time measurements of two observers are related by the Galilean transformations. Thus, when a particular from of an equation is determined by one observer, the Galilean transformation can be applied to this form to determine the form for the other observer. If both forms are the same, the equation is invariant under the Galilean transformations.

Solved problems

(1) A passenger in a train moving at 30 m/s passes a man standing on a station platform at $t = t' = 0$. Twenty seconds after the train passes him, the man on the platform determines that a bird flying along the tracks in the same direction as the train is 800 m away. What are the coordinates of the bird as determined by the passenger?

P1

The coordinates assigned to the bird by man on the station platform are $(x, y, z, t) = (800m, 0, 0, 20s)$

The passenger measures the distance x' to the bird as

$$x' = x - vt = 800m - (30m/s)(20s) = 200m$$

Therefore the bird's coordinates as determined by the passenger are

$$(x', y', z', t') = (200m, 0, 0, 20s)$$

(2) Refer to problem (1) five seconds after making the first coordinate measurement, the man on the platform determines that the bird is 850m away. From these data find the velocity of the bird (assumed constant) as

determined by the man on the platform and by the passenger on the train.

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P2

The coordinates assigned to the bird at the second position by the man on the platform are.

$$\left(x_2,Y_2,Z_2,t_2\right)=\left(850m,0,0.25s\right)$$

Hence the velocity

$$u_x=\frac{x_2-x_1}{t_2-t_1}=\frac{850m-800m}{25s-20s}=+10m/s$$

The positive sign indicates the bird is flying in the positive x direction. The passenger finds that at the second position the distance x_2' to the bird is

$$x_2'=x_2-vt_2=850m-\left(30m/s\right)\left(25s\right)=100m$$

Thus $\left(x_2',y_2',z_2',t_2'\right)=\left(100m,0,0,25s\right)$ and the velocity u_x' of the bird as measured by the passenger is

$$u_x'=\frac{x_2'-x_1'}{t_2'-t_1'}=\frac{100m-200m}{25s-20s}=-20m/s$$

So that, as measured by the passenger, the bird is moving in the negative x'-direction. Note that this result is consistent with that obtained from the Galilean velocity transformation:

$$u_x'=u_x-v=10m/s-30m/s=-20m/s$$

(3) A train moving with a velocity of 60 mi/hr passes through a railroad station at 12:00. Twenty seconds later a bolt of lightning strikes the railroad tracks one mile from the station in the same direction that the train is moving. Find the coordinates of the lightning flash as measured by an observer at the station and by the engineer of the train.

Both observers measure the time coordinate as:

$$t = t' = (20s)\left(\frac{1hr}{3600s}\right) = \frac{1}{180}hr$$

The observer at the station measures the spatial coordinate to be $x = 1\,mi$ .the spatial coordinate as determined by the engineer of the train is

$$x' = x - vt = 1\,mi - (60\,mi/hr)\left(\frac{1}{180}hr\right)$$

$$= \frac{2}{3}mi$$

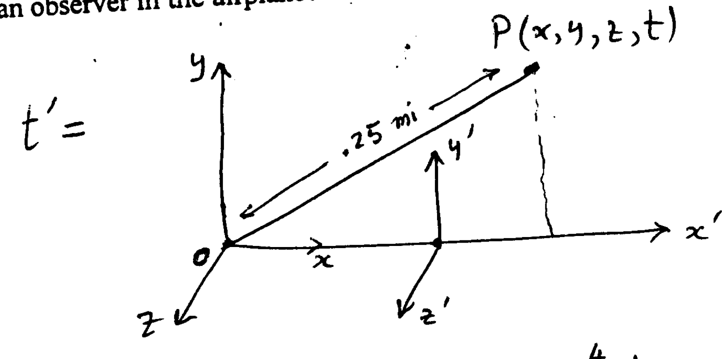
$$x' = \beta(x - vt) \qquad \beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t' = \beta\left(t - \frac{v}{c^2}x\right)$$

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(4) A hunter on the ground fires a bullet in the northeast direction which strikes a deer 0.25 miles from the hunter. The bullet travels with a speed of 1800 mi/hr. at the instant when the bullet is fired, an airplane is directly over the hunter at an altitude of one mile and is traveling due east with a velocity of 600 mi/hr. When the bullet strikes the deer, What are the coordinates as determined by an observer in the airplane?

$t' =$



$$t' = t = \frac{0.25 \ mi}{1800 \ mi/hr} = 1.39 \times 10^{-4} \ hr$$

$$(x, y, z, t) = (.25 \cos 45, .25 \sin 45, 0, t)$$

$$x' = x - vt = \frac{0.25}{\sqrt{2}} - 600 \times 1.39 \times 10^{-4} = 0.094 m$$

$y' =$

$z' =$

(5) An observer, at rest with respect to the ground, observes the following collision. A particle of mass $m_1 = 3kg$ moving with velocity $u_1 = 4m/s$ along the x-axis approaches a second particle of mass $m_2 = 1kg$ moving with velocity $u_2 = -3m/s$ along the x-axis. After a head-on collision the ground observer finds that m_2 has velocity $u_2^* = 3m/s$ along the x-axis. Find the velocity u_1^* of m_1 after the collision.

$$\textit{initial momentum} = \textit{final momentum}$$

$$m_1u_1 + m_2u_2 = m_1u_1^* + m_2u_2^*$$

$$(3)(4)+(1)(-3)=(3)u_1^* +(1)(3)$$

$$9=(3)u_1^* +3$$

Solving: $u_1^* = 2m/s.$

(6) A second observer, o' who is walking with a velocity of 2m/s relative to the ground along the x-axis observes the collision described in problem (5). What are the system moment before and after the collision as determined by him?

Using the Galilean velocity transformations, (complete)

$$u_1' = u_1 - v = 4 - 2 = 2$$

$$u_2' = u_2 - v = -3 - 2 = -5$$

$$u_1^{*\prime} = u_1^* - v = 2 - 2 = 0$$

$$u_2^{*\prime} = u_2^* - v = 3 - 2 = 1$$

$$m_1u_1' + m_2u_2' = m_1u_1^{*\prime} + m_2u_2^{*\prime} = 1$$

(7) An open car traveling at 100ft/s has a boy in it who throws a ball upward with a velocity of 20ft/s. write the equation of motion (giving position as a function of time) for the ball as seen by (a) the boy, (b) an observer stationary on the road.

(a) for the boy in the car the ball travels straight up and down, so

$$y' = v_o t' + \frac{1}{2} a t'^2 = (20)t' + \frac{1}{2}(-32)t'^2 = 20t' - 16t'^2$$
$$x' = z' = 0$$

(b) For the stationary observer, one obtains from the Galilean transformations.

$$t = t'$$
$$x = x' + vt' = 0 + 100t$$
$$y = y' = 20t - 16t^2$$
$$z = z' = 0$$

(8) Show that the electromagnetic wave equation.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0$$

is not invariant under the Galilean transformations.

The equation will be invariant if it retains the same form when expressed in terms of the new variables (x', y', z', t'). We first find from the Galilean transformation that

$$\frac{\partial x'}{\partial x}=1 \qquad \frac{\partial x'}{\partial t}=-v \qquad \frac{\partial t'}{\partial t}=\frac{\partial y'}{\partial y}=\frac{\partial z'}{\partial z}=1$$

$$\frac{\partial x'}{\partial y}=\frac{\partial x'}{\partial z}=\frac{\partial y'}{\partial x}=\frac{\partial t'}{\partial x}=....=0$$

From the chain rule and using the above results we have

$$\frac{\partial \varphi}{\partial x}=\frac{\partial \varphi}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial \varphi}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial \varphi}{\partial z'}\frac{\partial z'}{\partial x}+\frac{\partial \varphi}{\partial t'}\frac{\partial t'}{\partial x}=\frac{\partial \varphi}{\partial x'}$$

and

$$\frac{\partial^2 \varphi}{\partial x^2}=\frac{\partial^2 \varphi}{\partial x'^2}$$

Similarly,

$$\frac{\partial^2 \varphi}{\partial y^2}=\frac{\partial^2 \varphi}{\partial y'^2} \qquad \frac{\partial^2 \varphi}{\partial z^2}=\frac{\partial^2 \varphi}{\partial z'^2}$$

Moreover,

$$\frac{\partial \varphi}{\partial t}=-v\frac{\partial \varphi}{\partial x'}+\frac{\partial \varphi}{\partial t'}$$

$$\frac{\partial^2 \varphi}{\partial t^2}=\frac{\partial^2 \varphi}{\partial t'^2}-2v\frac{\partial^2 \varphi}{\partial x'\partial t'}+v^2\frac{\partial^2 \varphi}{\partial x'^2}$$

Substituting these expressions in the wave equation gives

$$\frac{\partial^2\varphi}{\partial x'^2}+\frac{\partial^2\varphi}{\partial y'^2}+\frac{\partial^2\varphi}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}$$
$$+\frac{1}{c^2}\left(2v\frac{\partial^2\varphi}{\partial x'\partial t'}-v^2\frac{\partial^2\varphi}{\partial x'^2}\right)=0$$

Therefore the wave equation is not invariant under the Galilean transformations, for the form of the equations has changed.

The electromagnetic wave equation follows from Maxwell's equations of electromagnetic theory.

By applying the procedure described here to Maxwell's equation, one fined that Maxwell's equation also are not invariant under Galilean transformations.

CHAPTER. 2

2- The lorentz coordinate transformations

(2.1) THE POSTULATES OF EINSTEIN:

Einstein's guiding idea, which he called the principle of relativity, was that all no accelerating observers should be treated equally in all respects. Even if they are moving (at constant velocity) relative to each other. This principle can be formalized as follows:

Postulate 1: the laws of physics are the same (invariant) for all inertial (no accelerating) observers.

Newton's laws of motion are in accord with the principle of relativity, but Maxwell's equations together with the Galilean transformations are in conflict with it. Einstein could see no reason for a basic difference between dynamical and electromagnetic laws. Hence his

Postulate 2: in vacuum the speed of light as measured by all inertial observers is

$$c = 1/\sqrt{\varepsilon_0 \mu_0} = 3\times 10^8 m/s$$

Independent of the motion of the source.

Postulate 2 requires that the Galilean coordinate transformations be replaced by the lorentz coordinate transformations. For the two observers of fig (1) these are:

$$x' = \frac{x - vt}{\sqrt{1-(v^2/c^2)}}, \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-(v^2/c^2)}}, \quad y' = y, \quad z' = z \qquad (2.1)$$

These equations can be inverted to give:

$$x = \frac{x' + vt'}{\sqrt{1-(v^2/c^2)}}, \quad t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1-(v^2/c^2)}}, \quad y = y', \quad z = z' \qquad (2.2)$$

In (2.1)and (2.2), v is the velocity of o' with respect to o along their common axis, v is positive if o' moves in the positive x-direction and negative if o' moves in the negative x-direction It has also been assumed that both origins coincide when the clocks are started, so that $t' = t = 0$ when $x' = x = 0$. Note that the inverse transformations can be obtained from the first set of transformations by interchanging primed and unprimed variables and letting v ----v . This is to be expected from postulate I, since both observers are completely equivalent and observer o moves with velocity -- v with respect to o'.

(2.2) THE CONSTANCY OF THE SPEED OF LIGHT:

Suppose that at the instant when o and o' pass each other

(at $t = t' = 0$), a light signal is sent from their common origin in the positive $x - x'$ direction. If o finds that the

signals spatial and time coordinates are related by $x = ct$, then, according to (2.1), o' will find that:

$$x' = \frac{x - vt}{\sqrt{1 - (v^2/c^2)}} = \frac{ct - vt}{\sqrt{1 - (v^2/c^2)}}$$

$$= \frac{ct[1 - (v/c)]}{\sqrt{[1 - (v/c)][1 + (v/c)]}}$$

$$= \left(\sqrt{\frac{1 - (v/c)}{1 + (v/c)}}\right)(ct)$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - (v^2/c^2)}}$$

$$= \frac{t - \frac{v}{c^2}ct}{\sqrt{1 - (v^2/c^2)}}$$

$$= \frac{t[1 - (v/c)]}{\sqrt{[1 - (v/c)][1 + (v/c)]}}$$

$$= \left(\sqrt{\frac{1 - (v/c)}{1 + (v/c)}}\right)(t)$$

Thus o' will find that $x' = ct'$, in agreement with the second postulate of Einstein. Note also that for this event $t' \neq t$, in definite disagreement with the Galilean assumption.

(2.3) GENERAL CONSIDERATIONS IN SOLVING PROBLEMS INVOLVING LORENTZ TRANSFORMATIONS.

When attacking any space-time problem, the key concept to keep in mind is that of "event." Most problems are concerned with two observers measuring the space and time coordinates of an event (or events). Thus each event has eight numbers associated with it: $(x, y, z.t)$, as assigned by o, and $(x', y', z'.t')$, as assigned by o'. The lorentz coordinate transformations express the relationships between these assignments.

Many times problems are concerned with the determination of the spatial interval and /or the time interval between two events. In this case a useful technique is to subtract from each other the appropriate lorentz transformations describing each event. For example. Suppose observer o' measures the time and spatial intervals between two events, A and B ,and it is desired to obtain the time interval between these same two events as measured by o. From (2.2) one obtains upon subtracting t_A from t_B

$$t_B - t_A = \frac{(t'_B - t'_A) + (v/c^2)(x'_B - x'_A)}{\sqrt{1 - (v^2/c^2)}} \quad \text{——} \quad (2.3)$$

Since all the quantities on the right-hand side of this equation are known, one can determine $t_B - t_A$.

(2.4) SIMULTANEITY.

Two events are simultaneous to an observer if he measures that the two events occur at the same time. With classical physics, when one observer determined that two events were simultaneous, then, since $t' = t$ from the Galilean transformations, every other observer would also find that the two events were simultaneous. In relativistic physics, on the other hand, two events that are simultaneous to one observer will, in general, not be simultaneous to another observer.

Suppose, for example, that events A and B are simultaneous as determined by o', so that $t'_A = t'_B$

According to (2.3), observer o will measure the time separation of these same two events as

$$t_B - t_A = \frac{\left(v/c^2\right)\left(x'_B - x'_A\right)}{\sqrt{1-\left(v^2/c^2\right)}}$$

If the two events occur at the same spatial location, so that $x'_B = x'_A$, then the two events will also be simultaneous as determined by o. But if $x'_B \neq x'_A$. o Will determine that the two events are not simultaneous.

Note that if the two events occur at the same spatial location, only one clock is needed by each observer to determine if the events are simultaneous. On the other hand, if the two events are separated spatially, then each observer needs two clocks, properly synchronized, to determine whether or not the two events are simultaneous.

① As measured by O a flashbulb goes off at
$x = 100km,\ y = 10km,\ z = 1km\ \ at\ \ t = 5\times10^{-4}s$.

What are the coordinates x', y', z', and, t' of this event as determined by a second observer, o', moving relative to o at $-0.8c$ along the common $x - x'$ axis?

From the lorentz transformations,

$$x' = \frac{x - vt}{\sqrt{1-(v^2/c^2)}}$$

$$= \frac{100-(-0.8\times3\times10^5)(5\times10^{-4})}{\sqrt{1-(0.8)^2}} = 367km$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-(v^2/c^2)}} = \frac{5\times10^{-4} - \frac{(-0.8)(100)}{(3\times10^5)}}{\sqrt{1-(0.8)^2}}$$

$$= 12.8\times10^{-4}s$$

$$y' = y = 10km$$
$$z' = z = 1km$$

② Suppose that a particle moves relative to o' with a constant velocity of $c/2$ in the $x'y'$-plane such that its trajectory makes an angle of $60°$ with the x'=axis. If the velocity of o' with respect to o is $0.6c$ along the $x - x'$ axis, find the equations of motion of the particle as determined by o.

The equations of motion as determined by o' are:

$$x' = u'_x t' = \frac{c}{2}\left(\cos 60^\circ\right)t'$$

$$y' = u'_y t' = \frac{c}{2}\left(\sin 60^\circ\right)t'$$

Substituting from (2.1) in the first expression, we obtain.

Complete the solution:

$$x' = \frac{x - vt}{\sqrt{}} \quad , t' = \frac{t - \frac{v}{c^2}x}{\sqrt{}}$$

$$\therefore \frac{x - vt}{\sqrt{}} = \left(\frac{c}{2}\cos 60\right)\frac{t - \frac{v}{c^2}x}{\sqrt{}}$$

$$x - 0.6ct = \frac{c}{4}\left(t - \frac{0.6}{c}x\right)$$

then $x = (0.74\,c)\,t \longrightarrow$

(3) A train 1/2 mile long(as measured by an observer on the train) is traveling at a speed of 100 mi/hr. two lightning bolts strike the ends of the train simultaneously as determined by an observer on the ground. What is the time separation as measured by an observer on the train?

We have

$$(100mi/hr)\left(\frac{1hr}{3600s}\right) = 2.78\times10^{-2}mi/s$$

Let events A and B be defined by the striking of each lighting blot. With O as the ground observer, we have from (2). bolt

$$t_B - t_A = \frac{(t'_B - t'_A) + \frac{v}{c^2}(x'_B - x'_A)}{\sqrt{1-(v^2/c^2)}}$$

$$o = \frac{(t'_B - t'_A) + \frac{2.78\times10^{-2}mi/s}{(1.86\times10^5 mi/s)^2}(0.5mi)}{\sqrt{1-(v^2/c^2)}}$$

Solving, $t'_B - t'_A = -4.02\times10^{-3}s$. The minus sign denotes that event A occurred after event B.

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(4) Observer o notes that two events are separated in space and time by $600m$ and $8\times10^{-7}s$. How fast must an observer o' be moving relative to o in order that the events be simultaneous to o'?

Subtracting two lorentz transformations, we obtain

$$t_2' - t_1' = \frac{(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)}{\sqrt{1 - (v^2/c^2)}}$$

$$O = \frac{8 \times 10^{-7} s - \frac{v}{c}\left(\frac{600m}{3 \times 10^8 m/s}\right)}{\sqrt{1 - (v^2/c^2)}}$$

Solving $v/c = 0.4$

(5) The space-time coordinates of two events as measured by O are :

$$x_1 = 6 \times 10^4 m, \quad y_1 = z_1 = 0m, \; t_1 = 2 \times 10^{-4} s$$

And

$$x_2 = 12 \times 10^4 m, \; y_2 = z_2 = 0m, \; t_2 = 1 \times 10^{-4} s$$

What must be the velocity of O' with respect to O if O' measures the two events to occur simultaneously?

$\frac{v}{c} = -\frac{1}{2}$

(6) Refer to problem (5) what is the spatial separation of the two events as measured by O' ?

Subtracting two lorentz transformations:

$$x_2' - x_1' = \frac{(x_2 - x_1) - v(t_2 - t_1)}{\sqrt{1-(v^2/c^2)}}$$

From problem (5)
$v/c = -1/2$ or $v = -1.5\times10^8 m/s$

$$x_2' - x_1' = \frac{(12\times10^4 - 6\times10^4) - (-1.5\times10^8)(1\times10^{-4} - 2\times10^{-4})}{\sqrt{1-(-0.5)^2}}$$

$$= 5.20\times10^4 m$$

X CHAPTER. 3

3- RELATIVISTIC LENGTH MEASUREMENTS

The definition of length:

If a body is at rest with respect to an observer, its length is determined by measuring the difference between the spatial coordinates of the endpoints of the body. Since the body is not moving, these measurements may be made at any time, and the length so determined is called the rest length or proper length of the body.

For a moving body, however, the procedure is more complicated, since the spatial coordinates of the endpoints of the body must be measured at the same time. The difference between these coordinates is then defined to be the length of the body.

Consider now a ruler, oriented along the $x-x'$ direction, that is at rest with respect to observer o.

We wish to determine how the length measurements of o and o' are related to each other when o' is moving relative to o with a velocity v in the $x-x'$ direction. Let the ends of the ruler be designated by A and B. From the inverse lorentz transformations we obtain

$$x'_B - x'_A = \frac{(x_B - x_A) + v(t_B - t_A)}{\sqrt{1-(v^2/c^2)}}$$

The difference $x'_B - x'_A = L'$ is the (proper) length of the ruler as measured by o'. If x_B and x_A are measured by o at

the same time, so that $t_B - t_A = 0$, then the difference $x_B - x_A = L$ will be the length of the ruler as measured by O. Thus we have

$$L = L' \sqrt{1-(v^2/c^2)}$$

Since $\sqrt{1-(v^2/c^2)} < 1$ we have $L < L'$ so that the length of the moving ruler will be measured by O to be contracted. This result is called the Lorentz- Fitzgerald contraction.

A warning!

It is essential to keep clear the distinction between the concepts of "spatial coordinate separation" and "length" . A common mistake in solving problems is simply to multiply or divide a given spatial interval by the term

$$\sqrt{1-(v^2/c^2)}.$$

This approach will work if one is concerned with finding the relations between lengths, where the concept of "length" is defined precisely above. However, if one is interested in the spatial interval between two events that do not occur simultaneously, then the answer is obtained from the subtraction technique of section (2), the correct answer will not be obtained by multiplying or dividing the original spatial separation by

$$\sqrt{1-(v^2/c^2)}.$$

(1) How fast does a rocket ship have to go for its length to be contracted to 99% of its rest length?

From the expression for length contraction,

$$\frac{L}{L'} = 0.99 = \sqrt{1-\left(v^2/c^2\right)} \quad or \quad v = 0.141c$$

(2) Calculate the Lorentz contraction of the earths diameter (in the plane of the ecliptic) as measured by an observer O' who is stationary with respect to the sun.

Taking the orbital velocity of the earth to be $3\times10^4\,m/s$ and the diameter of the earth as 7920 mi, the expression for the Lorentz contraction yields.

$$D = D'\sqrt{1-\left(v^2/c^2\right)} = \left(7.92\times10^3\right)\sqrt{1-\left(\frac{3\times10^4}{3\times10^8}\right)^2}$$

$$\approx \left(7.92\times10^3\right)\left(1-0.5\times10^{-8}\right)$$

Solving, $D' - D = 3.96\times10^{-5}\,mi = 2.51m$. It is seen that relativistic effects are very small at speeds that are normally encountered.

(3) A meter stick makes an angle of 30° with respect to the x'-axis of O'. What must be the value of v if the meter stick makes an angle of 45° with respect to the x-axis of O?

(4) A cube has a (proper) volume of $1000cm^3$. Find the volume as determined by an observer O' who moves at a velocity of $0.8c$ relative to the cube in a direction parallel to one edge.

The observer measures an edge of the cube parallel to the direction of motion to have the contracted length.

$$I'_x = I_x\sqrt{1-\left(v^2/c^2\right)}.$$

$$= (10cm)\sqrt{1-(0.8)^2} = 6cm$$

The lengths of the other edges are unchanged :

$$I'_y = I_y = I'_z = I_z = 10cm$$

Therefore,

$$V' = I'_x I'_y I'_z = (6cm)(10cm)(10cm) = 600cm^3$$

4-RELATIVISTIC TIME MEASUREMENTS

(4.1) Proper time.

If an observer, say O, determines that two events A and B occur at the same location, the time interval between these two events can be determined by O with a single clock. This time interval, $t_B - t_A = \Delta t$ as measured by O with his single clock, is called the proper time interval between the events.

(4.2) Time dilation.

Now consider the same two events A and B as viewed by a second observer, O', moving with a velocity v with respect to O. The second observer will necessarily determine that the two events occur at different location and will therefore have to use two different, properly synchronized clocks to determine the time separation $t'_B - t'_A = \Delta t'$ between A and B. To find the relationship between the time separations as measured by O and O' we subtract two of the lorentz time transformations, obtaining.

$$\Delta t' = \frac{\Delta t - \frac{v}{c^2}(x_B - x_A)}{\sqrt{1-(v^2/c^2)}}$$

Because O determines that the two events occur at the same location $x_B - x_A = 0$. Thus:

$$\Delta t' = \frac{\Delta t}{\sqrt{1-(v^2/c^2)}}$$

Since $\sqrt{1-v^2/c^2}<1, \Delta t'>\Delta t$ so that the time interval between the two events as measured by o' is dilated (enlarged).

In the above example the single clock was taken to be at rest with respect to o. The same result would obtain, however, if the single clock were taken to be at rest with respect to o'. Thus, in general, suppose a single clock advances through a time interval Δt_0. If this clock is moving with a velocity v with respect to an observer, he will determine that his two clocks — advance — through — a — time — interval Δt given by. $\Delta t' = \dfrac{\Delta t}{\sqrt{1-(v^2/c^2)}}$.

Solved problems

(1) Observers o and o' approach each other with a relative velocity of $0.6c$. If o measures the initial distance to o' to be $20m$, how much time will it take, as determined by o, before the two observers meet ?

We have:

$$\Delta t = \frac{distance}{velocity} = \frac{20m}{0.6 \times 3 \times 10^8 m/sec}$$

$$= 11.1 \times 10^{-8} s$$

(2) In problem (1) how much time will it take, as determined by o', before the two observers meet?

The two events under consideration are: (A) The position of o' when o makes his initial measurement, and (B). The coincidence of o and o'. Both of these events occur at the origin of o'.

Therefore the time lapse measured by o' is equal to the proper time between the two events. From the time dilation expression,

$$\Delta t' = (\Delta t)\sqrt{1-(v^2/c^2)} = (11.1\times10^{-8}s)\sqrt{1-(0.6)^2}$$
$$= 8.89\times10^{-8}s$$

This problem can also be solved by noting that the initial distance as determined by o' is related to the distance measured by o through the lorentz contraction:

$$L' = L_0\sqrt{1-(v^2/c^2)} = (20m)\sqrt{1-(0.6)^2} = 16m$$

Then

$$\Delta t' = \frac{L'}{v} = \frac{16m}{0.6\times3\times10^8 m/s} = 8.89\times10^{-8}s$$

(3) Pions have a half-life of $1.8\times10^{-8}s$. A pion beam leaves an accelerator at a speed of $0.8c$. Classically, what is the expected distance over which half the pions should decay?

We have:

$$\text{distance} = v\Delta t = \left(0.8\times 3\times 10^{8}\, m/s\right)\left(1.8\times 10^{-8}\, s\right)$$
$$= 4.32m$$

(4) Determine the answer to problem (3) relativistically.

The half-life of $1.8\times 10^{-8}\, s$ is determined by an observer at rest with respect to the pion beam. From the point of view of an observer in the laboratory, the half-life has been increased because of the time dilation, and is given by. (Complete)

CHAPTER. 5

5-RELATIVISTIC SPACE-TIME MEASUREMENTS.

Soled problems

(1) A meter stick moves with a velocity of $0.6c$ relative to you along the direction of its length. How long will it take for the meter stick to pass you?

The length of the meter stick as measured by you is obtained from the Lorentz contraction:

$$L' = L\sqrt{1-\left(v^2/c^2\right)} = (1m)\sqrt{1-(0.6)^2} = 0.8m$$

The time for the meter stick to pass you is then found from

$$\text{distance} = velocity \times time$$

$$0.8m = \left(0.6\times 3\times 10^8 m/s\right)\times \Delta t$$

$$\Delta t = 4.44\times 10^{-9}$$

(2) It takes 10^5 years for light to reach us from the most distant parts of our galaxy. Could a human travel there, at constant speed, in 50 years?

The distance traveled by light in 10^5 years is, according to an observer at rest with respect to the earth,

$$d = c(\Delta t) = 10^5 c$$

Where c is expressed in, say, mi/yr. if this observer now moves with constant speed v with respect to the earth,

the distance d that he has to travel is shortened according to the Lorentz contraction:

$$d' = d\sqrt{1-\left(v^2/c^2\right)} = \left(10^5 c\right)\sqrt{1-\left(v^2/c^2\right)}$$

The time interval available to travel this distance is 50 years, so that

$$v = \frac{d'}{\Delta t} = \frac{10^5 c\sqrt{1-\left(v^2/c^2\right)}}{50}$$

Solving, $\dfrac{v}{c} = \sqrt{1-\left(2.5\times10^{-7}\right)} \approx 0.999999875$

Therefore a human traveling at this speed will find that when he completes the trip he has aged 50 years.

✓ ③ Suppose an observer O determines that two events are separated by $3.6\times10^8\,m$ and occur $2s$ apart. What is the proper time interval between the occurrences of these two events?

There exists a second observer, O', moving relative to the first observer who will determine that the two events occur at the same spatial location. The proper time interval between the two events is the time interval measured by this observer. Denoting the two events by A and B, we obtain upon subtracting two lorentz transformations

$$x'_B - x'_A = \frac{(x_B - x_A) - v(t_B - t_A)}{\sqrt{1-(v^2/c^2)}}$$

$$0 = \frac{3.6\times10^8 m - v(2s)}{\sqrt{1-(v^2/c^2)}}$$

$$v = 1.8\times10^8 m/s = 0.6c$$

Again subtracting two lorentz transformations, we obtain the proper time interval as

$$t'_B - t'_A = \frac{(t_B - t_A) - \left(\frac{v}{c2}\right)(x_B - x_A)}{\sqrt{1-(v^2/c^2)}}$$

$$= \frac{2s - \frac{0.6\times3.6\times10^8 m/s}{3\times10^8 m/s}}{\sqrt{1-(0.6)^2}} = 1.6s$$

Another way to solve this problem is to use v and the time dilation expression:

$$\Delta t' = (\Delta t)\sqrt{1-(v^2/c^2)}$$

$$= (2s)\sqrt{1-(0.6)^2} = 1.6s$$

④ For observer o, two events are simultaneous and occur $600km$ apart. What is the time difference between these two events as determined by o', who measures their spatial separation to be $1200km$?

Let A and B designate the two events. Subtracting two lorentz transformations, we obtain

$$x'_B - x'_A = \frac{(x_B - x_A) - v(t_B - t_A)}{\sqrt{1-(v^2/c^2)}}$$

$$12\times10^5 m = \frac{6\times10^5 m - v(0)}{\sqrt{1-(v^2/c^2)}}$$

$$\therefore \frac{v}{c} = 0.866$$

Again subtracting two Lorentz transformations:

$$t'_B - t'_A = \frac{(t_B - t_A) - \frac{v}{c^2}(x_B - x_A)}{\sqrt{1-(v^2/c^2)}}$$

$$= \frac{0 - \frac{0.866(6\times10^5 m)}{3\times10^5 m/s}}{\sqrt{1-(0.866)^2}}$$

$$= -3.46\times10^{-3} s$$

The minus sign denotes that event A occurred after event B as determined by o'.

(5) A man in the back of a rocker shoots a high-speed bullet towards a target in the front of the rocket. The rocket is $60m$ long and the bullets speed is $0.8c$, both as measured by the man. Find the time that the bullet is in flight as measured by the man. We have :

$$\Delta t = \frac{distance}{velocity} = \frac{60m}{0.8 \times 3 \times 10^{8} m/s} 2.50 \times 10^{-7} s$$

(6) Refer to problem (5) if the rocket moves with a speed of $0.6c$ relative to the earth; find the time that the bullet is in flight as measured by an observer on the earth.

Subtracting two inverse lorentz transformations:

$$t_B - t_A = \frac{(t'_B - t'_A) + \frac{v}{c^2}(x'_B - x'_A)}{\sqrt{1-(v^2/c^2)}}$$

$$= \frac{2.5 \times 10^{-7} s + \frac{(0.6)(60m)}{3 \times 10^{8} m/s}}{\sqrt{1-(0.6)^2}} = 4.63 \times 10^{-7} s$$

(7) A rocket ship $90m$ long travels at a constant velocity of $0.8c$ relative to the ground. As the nose of the rocket ship passes a ground observer, the pilot in the nose of the ship shines a flashlight toward the tail of the ship. What time does the signal reach the tail of the ship as recorded by (a) the pilot, (b) The ground observer?

(8) The speed of a rocket with respect to a space station is $2.4\times10^8 m/s$, and observers o' and o in the rocket and the space station, respectively, synchronize their clocks in the usual fashion$\left(i.e.\ \ t=t'=0\ \ when\ \ x=x'=0\right)$. Suppose that o look at o's clock through a telescope. What time dose he see on o's clock when his own clock reads 30s?

Let event A and B be defined, respectively, by the emission of the light signal from o' and the reception of the same signal by o. Our problem is to find t'_A. Applying the inverse Lorentz transformations to event A, we obtain

$$t_A=\frac{t'_A+\left(v/c^2\right)x'_A}{\sqrt{1-\left(v^2/c^2\right)}}=\frac{t'_A+\left(v/c^2\right)(0)}{\sqrt{1-(0.8)^2}}=\frac{t'_A}{0.6}$$

$$x_A=\frac{x'_A+vt'_A}{\sqrt{1-\left(v^2/c^2\right)}}=\frac{0+\left(0.8\times3\times10^8 m/s\right)t'_A}{\sqrt{1-(0.8)^2}}$$

$$=\left(4.0\times10^8 m/s\right)t'_A$$

The light signal travels in the negative direction at speed c, so that.

$$x_B-x_A=-c\left(t_B-t_A\right)$$

Substituting from above.

$$0-\left(4.0\times10^8 m/s\right)t'_A=\left(-3\times10^8 m/s\right)\left(30s-\frac{t'_A}{0.6}\right)$$

Solving, $t'_A=10.0s$.

This result, and that of problem (9) point out the distinction between seeing an event and measuring the coordinates of the same event.

(9) Refer to problem (8) if O' looks at O's clock through a telescope, what time does his own clock read when he sees O's clock reading 30 s ?

(10) Show that the electromagnetic wave equation.

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-\left(\frac{1}{c^2}\right)\left(\frac{\partial^2\varphi}{\partial t^2}\right)=0$$

Is invariant under a Lorentz transformation.

The equation will be invariant if it retains the same from when expressed in terms of the new variables x', y', z', t'. To express the wave equation in terms of the primed variables we first find from the lorentz transformations that

$$\frac{\partial x'}{\partial x}=\frac{1}{\sqrt{1-(v^2/c^2)}} \qquad \frac{\partial x'}{\partial t}=-\frac{v}{\sqrt{1-(v^2/c^2)}}$$

$$\frac{\partial t'}{\partial x}=-\frac{(v/c^2)}{\sqrt{1-(v^2/c^2)}} \qquad \frac{\partial t'}{\partial t}=\frac{1}{\sqrt{1-(v^2/c^2)}}$$

$$\frac{\partial y'}{\partial y}=\frac{\partial z'}{\partial z}=1 \qquad \frac{\partial x'}{\partial y}=\frac{\partial x'}{\partial z}=\frac{\partial y'}{\partial x}=.....=0$$

From the chain rule and using the above results. We have

$$\frac{\partial\varphi}{\partial x}=\left(\frac{\partial\varphi}{\partial x'}\right)\left(\frac{\partial x'}{\partial x}\right)+\left(\frac{\partial\varphi}{\partial t'}\right)\left(\frac{\partial t'}{\partial x}\right)$$

$$=\left(\frac{1}{\sqrt{1-(v^2/c^2)}}\right)\left(\frac{\partial\varphi}{\partial x'}\right)+\left(\frac{-v/c^2}{\sqrt{1-(v^2/c^2)}}\right)\left(\frac{\partial\varphi}{\partial t'}\right)$$

Differentiating again with respect to x, we have

$$\frac{\partial^2\varphi}{\partial x^2}=\frac{1}{1-\left(\frac{v^2}{c^2}\right)}\left[\frac{\partial^2\varphi}{\partial x'^2}+\left(\frac{v^2}{c^4}\right)\left(\frac{\partial^2\varphi}{\partial t'^2}\right)-\left(\frac{2v}{c^2}\right)\left(\frac{\partial^2\varphi}{\partial x'\partial t'}\right)\right]$$

Similarly we have

$$\frac{\partial\varphi}{\partial t}=\left(\frac{-v}{\sqrt{1-\left(v^2/c^2\right)}}\right)\left(\frac{\partial\varphi}{\partial x'}\right)+\left(\frac{1}{\sqrt{1-\left(v^2/c^2\right)}}\right)\left(\frac{\partial\varphi}{\partial t'}\right)$$

$$\frac{\partial^2\varphi}{\partial t^2}=\frac{1}{1-\left(v^2/c^2\right)}\left(\frac{\partial^2\varphi}{\partial x'^2}+\frac{v^2\partial^2\varphi}{c^2\partial t'^2}-2v\frac{\partial^2\varphi}{\partial x'\partial t'}\right)$$

$$\frac{\partial^2\varphi}{\partial y^2}=\frac{\partial^2\varphi}{\partial y'^2}\qquad\frac{\partial^2\varphi}{\partial z^2}=\frac{\partial^2\varphi}{\partial z'^2}$$

Substituting these in the wave equation, we obtain

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-\left(\frac{1}{c^2}\right)\left(\frac{\partial^2\varphi}{\partial t^2}\right)$$
$$=\frac{\partial^2\varphi}{\partial x'^2}+\frac{\partial^2\varphi}{\partial y'^2}+\frac{\partial^2\varphi}{\partial z'^2}-\left(\frac{1}{c^2}\right)\left(\frac{\partial^2\varphi}{\partial t'^2}\right)$$

So that the equation is invariant under Lorentz transformation. Recall that the wave equation is not invariant under Galilean transformation.

6- Relativistic velocity transformations:

(6.1) the lorentz velocity transformations and the speed of light:

Consider now the experiment discussed in section 2.2 where a light signal is sent in the $x - x'$ direction from the common origin when o and o' pass each other at $t = t' = 0$. If o measures the signals velocity components to be $u_x = c . u_y = u_z = 0$. Then, by (6.1), o' will measure

$$u'_x = \frac{u_x - v}{1 - \left(v / c^2\right) u_x} = \frac{c - v}{1 - \left(v / c^2\right) c} = c$$

$$u'_y = u'_z = 0$$

Thus o' also determines that the light signal travels with speed c, in accord with the second postulate of Einstein.

(6.2) General considerations in solving velocity problems.

In velocity problems there are three objects involved : two observers, o and o', and a particle, p. The particle p has two velocities (and, hence, six numbers) associated with it. Its velocity with respect to o, $\left(u_x, u_y, u_z\right)$, and its velocity with respect to o', $\left(u'_x, u'_y, u'_z\right)$. The quantity v appearing in the velocity transformations is the velocity of o' with respect to o.

When attacking a velocity problem. One should first determine which objects in the problem are to be identified with o, o', and p. Sometimes this identification is dictated, other times the identification can be made arbitrarily (see, for example, problem 1). Once the identification has been made. One then uses the appropriate Lorentz velocity transformations to achieve the answer. In dealing with velocity problems the best way to avoid mistakes is not to forget the phrase "with respect to" the phrase" velocity of an object" is meaningless (both classically and relativistically) because a velocity is always measured with respect to something.

Solved problems

(1) Rocket A travels to the right and rocket B travels to the left, with velocities $0.8c$ and $0.6c$, respectively, relative to the earth. What is the velocity of rocket A measured from rocket B?

Let observers o, o' and the particle be associated with the earth, rocket B, and rocket A, respectively. Then

$$u'_x = \frac{u_x - v}{1-\left(v/c^2\right)u_x} = \frac{0.8c-(-0.6c)}{1-\frac{(-0.6c)(0.8c)}{c^2}} = 0.946c$$

The problem can also be solved with other associations. For example, let observers o, o' and the particle be associated with rocket A, rocket B, and the earth, respectively. Then

$$u'_x = \frac{u_x - v}{1-\left(v/c^2\right)u_x} \quad or \quad 0.6c = \frac{-0.8c - v}{1-\left(v/c^2\right)(0.8c)}$$

Solving $v = -0.946c$, which agrees with the above answer. (the minus sign appears because v is the velocity of O' with respect to O, which, with the present association, is the velocity of recket B with respect to rocket A.)

② Repeat problem (1) if rocket A travels with a velocity of $0.8c$ in the $+y$-direction relative to the earth. (Rocket B still travels in the $-x$-direction.)

Let observers O, O' and the particle be associated with the earth, rocket B, and rocket A, respectively. Then

$$u'_x = \frac{u_x - v}{1-\left(v/c^2\right)u_x} = \frac{0-(-0.6c)}{1-0} = 0.6c$$

$$u'_y = \frac{u_y\sqrt{1-\left(v^2/c^2\right)}}{1-\left(v/c^2\right)u_x} = \frac{(0.8c)\sqrt{1-(0.6)^2}}{1-0}$$

$$= 0.64c$$

Which give the magnitude and direction of the desired velocity as

$$u' = \sqrt{u'^2_x + u'^2_y} = \sqrt{(0.6c)^2 + (0.64c)^2} = 0.88c$$

And

$$\tan\varphi' = \frac{u'_y}{u'_x} = \frac{0.64c}{0.60c} = 1.07 \qquad or \qquad \varphi' = 46.8^\circ$$

- (3) A particle moves with a speed of $0.8c$ at an angle of 30° to the x-axis, as determined by o. What is the velocity of the particle as determined by a second observer, o', moving with a speed of $-0.6c$ along the common $x-x'$-axis ?

$u_x = (0.8c)\cos 30 = 0.693c$

$u_y = (0.8c)\sin 30 = 0.4c$

$$u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}$$

$$u'_y = \frac{u_y}{\beta\left(1 - \frac{v}{c^2}u_x\right)}$$

(4) At $t = 0$ observer o emits a photon traveling in a direction of 60° with the x-axis. A second observer, o', travels with a speed of $0.6c$ along the common $x - x'$ axis. What angle does the photon make with the x'-axis of o'?

$u_x = c \cos 60 = 0.5c$ ⟶

$u_y = c \sin 60 = 0.866\, c$ ⟶

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$$u'_x = \frac{u_x - V}{1 - \frac{V}{c^2} u_x}$$

$$u'_x = u_x - V$$

$$u'_y = \frac{u_y}{\beta\left(1 - \frac{V}{c^2} u_x\right)}$$

$$u'_y = u_y$$

CHAPTER. 7

7- Mass, energy and momentum in relativity.

(7.1) The need to redefine classical momentum:

One of the major developments to come out of the special theory of relativity is that the mass of a body will vary with its velocity. A heuristic argument for this variation can be given as follows. Consider a ballistics experiment where an observer. Say o'. Fires a bullet in the y'-direction into a block that cannot move relative to him. It is reasonable to suppose that the amount the bullet penetrates into the block is determined by the y'-component of the momentum of the bullet. Given by $p'_y = m'u'_y$, where m' is the mass of the bullet as measured by o'.

Now consider the same experiment from the point of view of observer o who sees observer o' moving in the common $x - x'$ direction with a velocity v. Since the tunnel left by the bullet is at right angles to the direction of relative motion. o will agree with o' as to the distance that the bullet penetrates into the block, and therefore would expect to find the same value as o' for the y-component of the bullets momentum.

As determined by $o, p_y = mu_y$, where m is the mass of the bullet as measured by o. From the lorentz velocity transformations we find, since $u'_x = 0$. That

$$u_y = \frac{u'_y\sqrt{1-(v^2/c^2)}}{1+(v/c^2)u'_x} = u'_y\sqrt{1-(v^2/c^2)}$$

So that $p_y = mu'_y\sqrt{1-(v^2/c^2)}$. Since from above $p'_y = m'u'_y$.

It is seen that if both observers assign the same mass to the bullet, so that $m' = m$, they will find $p'_y \neq p_y$, contrary to what is expected.

(7.2) the variation of mass with velocity:

At this point we have two choices. We can assume that momentum principles-in particular. Conservation of momentum- do not apply at large velocity, or, we can look for a way to redefine the momentum of a body in order to make momentum principles applicable to special relativity. The latter alternative was chosen by Einstein. He showed that all observers will find classical momentum principles to hold if the mass m of a body varies with its speed u according to

$$m = \frac{m_0}{\sqrt{1-(u^2/c^2)}}$$

Where m_0, the rest mass is the mass of the body measured when it is at rest with respect to the observer. See problem 7.1

(7.3) Newton's second law in relativity:

The classical expression of Newton's second law is that the net force on a body is equal to the rate of change of the body's momentum. To include relativistic effects, allowance must be made for the fact that the mass of a body varies with its velocity. Thus the relativistic generalization of Newton's second law is

$$F=\frac{dp}{dt}=\frac{d}{dt}\left[\frac{m_0 u}{\sqrt{1-\left(u^2/c^2\right)}}\right]=\frac{d}{dt}(mu)$$

(7.4) Mass and energy relationship: $E=mc^2$

In relativistic mechanics, as in classical mechanics, the kinetic energy, K, of a body is equal to the work done by an external force in increasing the speed of the body from zero to some value u, i.e.

$$K=\int_{u=0}^{u=u} F\cdot ds$$

Using Newton's second law. $F=d(mu)/dt$, One finds (problem 7.21) that this expression reduces to

$$K=mc^2-m_0c^2$$

The kinetic energy. K, represents the difference between the total energy, E, of the moving particle and the rest energy, E_0, of the particle when at rest, so that

$$E-E_0=mc^2-m_0c^2$$

If the rest energy is chosen so that $E_0 = m_0c^2$, we obtain Einstein's famous relation $E = mc^2$

Which shows the equivalence of mass and energy. Thus, even when a body is at rest it still has an energy content given by $E_0 = m_0c^2$, so that in principle a massive body can be completely converted into another, more familiar, from of energy.

(7.5) Momentum and energy relationship:

Since momentum is conserved, but not velocity, it is often useful to express the energy of a body in terms of its momentum rather than its velocity. To this end, if the expression

$$m = \frac{m_0}{\sqrt{1-(u^2/c^2)}}$$

Is squared and both sides are multiplied by $c^4\left[1-(u^2/c^2)\right]$. One obtains

$$m^2c^2 - m^2u^2c^2 = m_0^2c^4$$

Using the results $E = mc^2$. $E_0 = m_0c^2$. And $|p| = mu$. We find the desired relationship between E and p to be

$$E^2 = (pc)^2 + E_0^2 \quad \text{or} \quad (K + m_0c^2)^2 = (pc)^2 + (m_0c^2)^2$$

(7.6) Units for energy and momentum:

The electron volt (eV) is the kinetic energy of a body whose charge equals the charge of an electron, after it moves through a potential difference of one volt.

$$1eV = (1.602\times10^{-19}c)(1V) = 1.602\times10^{-19}j$$

$$1MeV = 10^6 eV \qquad 1GeV = 10^9 eV$$

The relationship $1.602\times10^{-19}j \doteq 1eV$ can be looked at as conversion factor between two different units of energy.

The standard units for momentum are $kg\cdot m/s$. In relativistic calculations, however, units of MeV/c are frequently used for momentum. These units arise from the energy-momentum expression

$$p = \frac{\sqrt{E^2 - E_0^2}}{c}$$

The conversion factor is

$$1\frac{MeV}{c} = 0.534\times10^{-21}\frac{kg\cdot m}{s}$$

(7.7) General considerations in solving mass-energy problems:

A common mistake in solving mass-energy problem is to use the wrong expression for the kinetic energy. Thus

$$K \neq \tfrac{1}{2}m_0u^2 \quad and \quad K \neq \tfrac{1}{2}mu^2$$

The correct expression for the kinetic energy is

$$K = (m - m_0)c^2$$

Likewise, concerning the momentum, we note that

$$p \neq m_0 u$$

Solved problems

(1) From the rest masses calculate the rest energy of an electron in joules and electron-volts.

We have:

$$E_0 = m_0 c^2 = (9.109 \times 10^{-31})(2.998 \times 10^8)^2$$

$$= 8.187 \times 10^{-14}\, j, and$$

$$(8.187 \times 10^{-14}\, j)\left(\frac{1eV}{1.602 \times 10^{-19}\, j}\right)\left(\frac{1MeV}{10^6 eV}\right)$$

$$= 0.511 MeV$$

(2) A body at rest spontaneously breaks up into two parts which move in opposite directions. The parts have rest masses of $3kg$ and $5.33kg$ and respective speeds of $0.8c$ and $0.6c$. Find the rest mass of the original body.

since $E_{initial} = E_{final}$,

$$m_0 c^2 = \frac{m_{01} c^2}{\sqrt{1 - (v_1^2 / c^2)}} + \frac{m_{02} c^2}{\sqrt{1 - (v_2^2 / c^2)}}$$

$$m_0c^2 = \frac{(3kg)c^2}{\sqrt{1-(0.8)^2}} + \frac{(5.33kg)c^2}{\sqrt{1-(0.6)^2}}$$

$$m_0 = 11.66kg$$

(3) Calculate the momentum of a $1MeV$ electron

$$E^2 = (pc)^2 + E_0^2$$

$$(1MeV + 0.511MeV)^2 = (pc)^2 + (0.511MeV)^2$$

$$p = 1.42 MeV/c$$

(4) Calculate the kinetic energy of an electron whose momentum is $2MeV/c$.

$$E^2 = (pc)^2 + E_0^2$$

$$(K + 0.511MeV)^2 = \left(\frac{2MeV}{c}\right)^2 + (0.511MeV)^2$$

$$K = 1.55\ MeV$$

(5) Calculate the velocity of an electron whose kinetic energy is $2MeV$.

$$K = \frac{m_0c^2}{\sqrt{1-(v^2/c^2)}} - m_0c^2$$

$$2MeV = \frac{0.511MeV}{\sqrt{1-(v^2/c^2)}} - 0.511MeV$$

$$v = 0.98c$$

(6) Calculate the momentum of an electron whose velocity is $0.8c$. Consider $m_0c^2 = 0.511$ MeV

$$P = mv = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{v}{c^2}$$

$$P = \frac{0.511}{\sqrt{1-(0.8)^2}} \cdot \frac{0.8}{c} =$$

(7) An electron is accelerated from rest to a velocity of $0.5c$. Calculate its change in energy.

$$\text{change in energy} = \frac{m_0c^2}{\sqrt{1-(v^2/c^2)}} - m_0c^2$$

$$= \frac{0.511 MeV}{\sqrt{1-(0.5)^2}} - 0.511 MeV = 0.079 MeV$$

(8) At what fraction of the speed of light must a particle move so that its kinetic energy is double its rest energy?

$\because T = mc^2 - m_o c^2$

$T = mc^2 - m_o c^2 = 2 m_o c^2$

$\therefore m_o c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) = 2 m_o c^2$

$\frac{1}{\sqrt{}} = 3$

$\therefore v = 0.943\, c$

(9) An electrons velocity is $5\times10^{7}\,m/s$. How much energy is needed to double the speed?

$$E = mc^2$$

$$E_i = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{.511\ \text{Mev}}{\sqrt{1-\left(\frac{0.5\times10^8}{3\times10^8}\right)^2}} = 0.518\ \text{Mev}$$

$$E_f = \frac{m_0 c^2}{\sqrt{1-\frac{(2v)^2}{c^2}}} = \frac{0.511}{\sqrt{1-\left(\frac{1\times10^8}{3\times10^8}\right)^2}} = 0.542\ \text{Mev}$$

$$E_f - E_i = 0.024\ \text{Mev}$$

Consider (Given), the rest energy of the electron $m_0c^2 = 0.511$ Mev

(10) Two identical bodies, each with rest mass m_0.
Approach each other with equal velocities u, collide, and stick together in a perfectly inelastic collision. Determine the rest mass of the composite body.

Since the initial velocities are equal in magnitude. And the final momentum must be zero.

$$\sin ce E_{initial} = E_{final}$$

$$\frac{2m_0c^2}{\sqrt{1-\left(u^2/c^2\right)}} = m_0c^2$$

$$M_0 = \frac{2m_0}{\sqrt{1-\left(u^2/c^2\right)}} > 2m_0$$

(11) What is the rest mass of the composite body in problem 10 as determined by an observer who is at rest with respect to one of the initial bodies?